# 8 Problem VDPOL

## 8.1 General information

The problem consists of a second order differential equation rewritten to first order form, thus providing a system of ordinary differential equations of dimension 2. It was proposed by B. van der Pol in the 1920's [vdP20], [vdP26]. The INdAM-Bari Test Set group contributed this problem to the test set. Most of the documentation about this problem has been retrieved from [EP02]. The software part of the problem is in the files vdpol.f and vdpolm.f available at [MM08].

## 8.2 Mathematical description of the problem

The problem is of the form

$$z'' \quad = \quad f(z, z'), \quad z(0) = z_0 \quad z'(0) = z'_0,$$

with

$$z \in \mathbb{R}, \quad t \in [0, T],$$

where the function $f$ is given by

$$f(z, z') = \mu(1 - z^2)z' - z, \quad \mu > 0. \tag{II.8.1}$$

We write this problem to first order form by defining $y_1 = z$ and $y_2 = z'$, yelding a system of 2 nonlinear differential equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ f(y_1, y_2) \end{pmatrix} \tag{II.8.2}$$

with

$$(y_1, y_2)^T \in \mathbb{R}^2, \quad t \in [0, T].$$

A rescaling of the solutions of (II.8.2) results in the following formulation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} y_2 \\ \widetilde{f}(y_1, y_2) \end{pmatrix}, \tag{II.8.3}$$

where

$$\widetilde{f}(y_1, y_2) = ((1 - y_1^2)y_2 - y_1)/\epsilon, \quad \epsilon > 0.$$

Problem (II.8.2) will be referred to as $\text{vdpol}_\mu$ and problem (II.8.3) as $\text{vdpol}_\epsilon$. The initial values are

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} z_0 \\ z'_0 \end{pmatrix} \quad \text{where} \quad \begin{cases} z_0 = 2 \\ z'_0 = 0 \end{cases}.$$

## 8.3 Origin of the problem

The VDPOL problem originates from electronics and describes the behaviour of nonlinear vacuum tube circuits. The circuit scheme, designed by Balthazar van der Pol in the 1920's, is given in Figure II.8.1. This is an RLC loop, but with the passive resistor of Ohm's Law replaced by an active element which would pump energy into the circuit whenever the amplitude of the current falls below a certain level. In the 1920's this active element was an array of vacuum tubes, now it is a semiconductor device. The voltage drop at the semiconductor (which used to be RI) is given by a nonlinear function $f(I)$ of

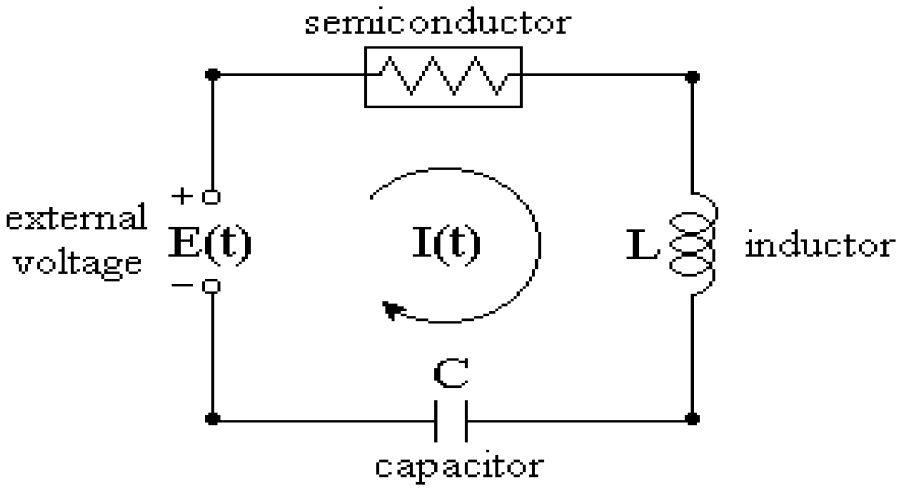


FIGURE II.8.1: *Negative resistance oscillatory circuit*

the current I. If we substitute $f(I)$ for $RI$ in the standard RLC-circuit equation $LI'' + RI' + I/C = 0$, the current in the circuit turns out to be modeled by

$$LI'' + f'(I)I' + I/C = 0. \tag{II.8.4}$$

In a 1924 study of oscillator circuits in early commercial radios (at Philips research laboratories), B. van der Pol assumed the voltage drop to be represented by the nonlinear function $f(I) = bI^3 - aI$, which with equation (II.8.4) leads to

$$LI'' + (3bI^2 - a)I' + I/C = 0. \tag{II.8.5}$$

This equation is also closely related to the equation introduced by the British mathematical physicist Lord Rayleigh (John William Strutt, 1842 - 1919) to model the oscillations of a clarinet reed. For more details see [EP02].

If we denote by $\tau$ the time variable in Eq. (II.8.5) and make the substitutions $I = pz$ and $t = \tau/\sqrt{LC}$, the result is

$$\frac{d^2z}{dt^2} + (3bp^2z^2 - a)\sqrt{\frac{C}{L}}\frac{dz}{dt} + z = 0.$$

With $p = \sqrt{a/(3b)}$ and $\mu = a\sqrt{C/L}$ this gives the standard form

$$z'' + \mu(z^2 - 1)z' + z = 0$$

of the *van der Pol's equation*.

The van der Pol equation is often used as a test problem for ODEs solvers. It has two periodic solutions, the constant solution, $z(t) \equiv 0$, that is unstable, and the nontrivial periodic solution (roughly corresponding to the initial conditions $z(0) = 2,\ z'(0) = 0$), that is named 'limit cycle' because all the other nontrivial solutions converge to this one as $t \to \infty$.

This qualitative behavior is well shown in the phase plane plot in Figure II.8.2 (for $\mu = 2$), where outward and inward spiral trajectories converge to the limit cycle (the closed curve).

The parameter $\mu > 0$ weights the importance of the nonlinear part of the equation. When $\mu$ is 'large' the approach to the limit cycle is quite rapid (see Figure II.8.3 for $\mu = 10^3$) and the van der Pol equation is more interesting because of the non negligible influence of the nonlinear term. From

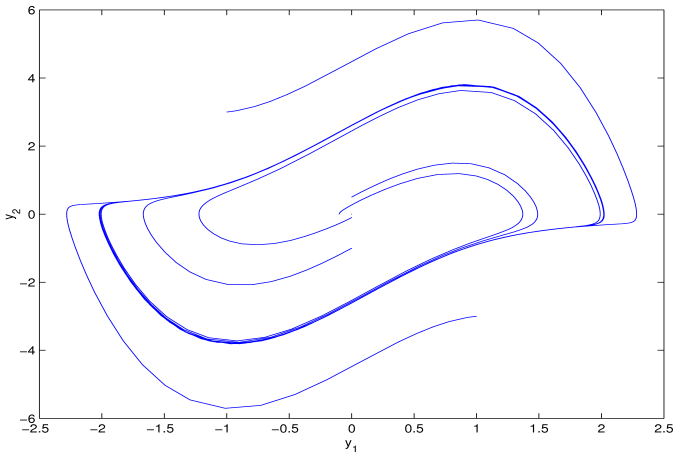


FIGURE II.8.2: *Limit cycle for* $\mu = 2$

an analysis of the behavior of the limit cycle [Sha94] it turns out that it can be described in terms of portions where the solution components change slowly and the problem is quite stiff, alternating with regions of very sharp change (quasi-discontinuities) where it is non-stiff. Thus, the problem switches from stiff to non stiff with a very sharp changing solution that makes the equation quite challenging for ODEs solvers.

The van der Pol equation may be treated in different ways, the most straightforward is to split the equation into a system of two first order differential equations as in (II.8.2). Note that if the second of the equations is divided by $\mu$ we get an equation that has the character of a singular perturbation problem. Several other approaches may show other aspects on the nature of this problem. For example Hairer and Wanner [HW96] introduce the following scaling transformation of (II.8.2) to make the steady-state approximation independent of $\mu$:

$$x = t/\mu, \quad w_1(x) = y_1(t), \quad w_2(x) = \mu y_2(t)$$

Substituting in (II.8.2) and using again $y$ for $w$ and $t$ for $x$, the equation (II.8.3) is obtained with $\varepsilon = 1/\mu^2$. The scaled version (II.8.3) has the advantage that a small interval independent of the parameter value can be considered to track at least one period of the solution.

## 8.4 Numerical solution of the problem

### 8.4.1 vdpol$_\mu$ with $\mu = 10^3$ and $t \in [0, 2\mu]$

Tables II.8.1, II.8.2 and Figures II.8.4, II.8.6–II.8.9 present the reference solution at the end of the integration interval, the run characteristics, the behavior of the first component of the solution over the integration interval and the work-precision diagrams, respectively. The reference solution was computed by RADAU on an Alphaserver DS20E, with a 667 MHz EV67 processor, using double precision `work(1) = uround` $= 1.01 \cdot 10^{-19}$, `rtol = atol = h0` $= 1.1 \cdot 10^{-18}$. For the work-precision diagrams, we used: rtol $= 10^{-(4+m/4)}$, $m = 0, 1, \ldots, 32$; atol = rtol; h0 $= 10^{-2} \cdot$ rtol for BIMD, GAMD, MEBDFDAE, MEBDFI, RADAU and RADAU5.

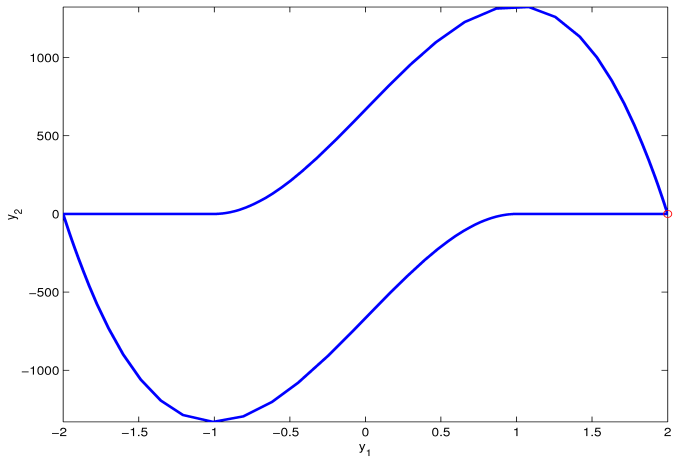


FIGURE II.8.3: *Limit cycle for* $\mu = 10^3$

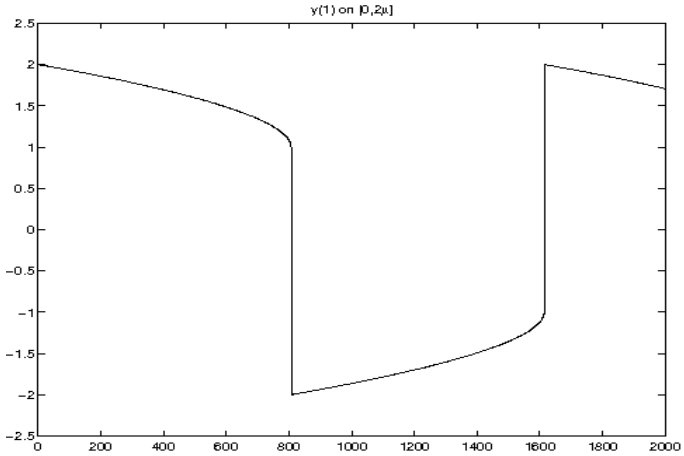


FIGURE II.8.4: *Behavior of the solution component* $y_1$ *over the integration interval*

### 8.4.2 vdpol$_\epsilon$ with $\epsilon = 10^{-6}$ and $t \in [0, 2]$

Tables II.8.3, II.8.4 and Figures II.8.5, II.8.10–II.8.13 present the reference solution at the end of the integration interval, the run characteristics, the behavior of the first component of the solution over the integration interval and the work-precision diagrams, respectively. The reference solution was computed by RADAU on an Alphaserver DS20E, with a 667 MHz EV67 processor, using double precision `work(1) = uround` $= 1.01 \cdot 10^{-19}$, `rtol = atol = h0` $= 1.1 \cdot 10^{-18}$. For the work-precision

TABLE II.8.1: *Reference solution at the end of the integration interval.*

| | |
|---|---|
| $y_1$ | $0.1706167732170469 \cdot 10^{1}$ |
| $y_2$ | $-0.8928097010248125 \cdot 10^{-3}$ |

TABLE II.8.2: *Run characteristics.*

| solver | rtol | atol | h0 | mescd | scd | steps | accept | #f | #Jac | #LU | CPU |
|---|---|---|---|---|---|---|---|---|---|---|---|
| BIMD | $10^{-4}$ | $10^{-4}$ | $10^{-6}$ | 4.05 | 3.57 | 133 | 112 | 2801 | 104 | 133 | 0.0020 |
| | $10^{-7}$ | $10^{-7}$ | $10^{-9}$ | 9.18 | 8.80 | 224 | 219 | 5072 | 209 | 224 | 0.0029 |
| | $10^{-10}$ | $10^{-10}$ | $10^{-12}$ | 11.17 | 10.32 | 250 | 248 | 10151 | 237 | 250 | 0.0078 |
| DDASSL | $10^{-4}$ | $10^{-4}$ | | 2.88 | 2.37 | 549 | 507 | 940 | 122 | | 0.0020 |
| | $10^{-7}$ | $10^{-7}$ | | 5.57 | 5.06 | 1342 | 1296 | 1980 | 129 | | 0.0049 |
| | $10^{-10}$ | $10^{-10}$ | | 8.25 | 7.73 | 4484 | 4445 | 5943 | 168 | | 0.0166 |
| GAMD | $10^{-4}$ | $10^{-4}$ | $10^{-6}$ | 4.86 | 4.30 | 129 | 90 | 5133 | 91 | 129 | 0.0039 |
| | $10^{-7}$ | $10^{-7}$ | $10^{-9}$ | 7.55 | 6.71 | 173 | 137 | 9422 | 141 | 173 | 0.0078 |
| | $10^{-10}$ | $10^{-10}$ | $10^{-12}$ | 9.53 | 9.17 | 235 | 197 | 16067 | 201 | 235 | 0.0127 |
| MEBDFI | $10^{-4}$ | $10^{-4}$ | $10^{-6}$ | 3.31 | 2.86 | 477 | 435 | 1761 | 83 | 83 | 0.0029 |
| | $10^{-7}$ | $10^{-7}$ | $10^{-9}$ | 6.11 | 5.60 | 1134 | 1083 | 3818 | 118 | 118 | 0.0059 |
| | $10^{-10}$ | $10^{-10}$ | $10^{-12}$ | 9.06 | 8.55 | 2135 | 2098 | 7215 | 208 | 208 | 0.0107 |
| PSIDE-1 | $10^{-4}$ | $10^{-4}$ | | 6.42 | 3.43 | 181 | 149 | 2811 | 57 | 648 | 0.0029 |
| | $10^{-7}$ | $10^{-7}$ | | 7.20 | 6.32 | 310 | 293 | 6141 | 52 | 756 | 0.0059 |
| | $10^{-10}$ | $10^{-10}$ | | 9.99 | 9.14 | 1000 | 990 | 15536 | 109 | 1156 | 0.0156 |
| RADAU | $10^{-4}$ | $10^{-4}$ | $10^{-6}$ | 4.48 | 4.28 | 210 | 172 | 1822 | 144 | 208 | 0.0010 |
| | $10^{-7}$ | $10^{-7}$ | $10^{-9}$ | 8.56 | 8.18 | 240 | 222 | 3508 | 187 | 238 | 0.0020 |
| | $10^{-10}$ | $10^{-10}$ | $10^{-12}$ | 10.63 | 9.24 | 209 | 176 | 6240 | 130 | 207 | 0.0039 |
| VODE | $10^{-4}$ | $10^{-4}$ | | 3.29 | 3.08 | 545 | 487 | 779 | 19 | 117 | 0.0020 |
| | $10^{-7}$ | $10^{-7}$ | | 5.20 | 4.73 | 1614 | 1502 | 2145 | 30 | 223 | 0.0049 |
| | $10^{-10}$ | $10^{-10}$ | | 7.49 | 7.07 | 4350 | 4120 | 5266 | 72 | 516 | 0.0146 |

diagrams, we used: rtol $= 10^{-(4+m/4)}$, $m = 0, 1, \ldots, 32$; atol = rtol; h0 $= 10^{-2} \cdot$ rtol for GAMD, MEBDFDAE, MEBDFI, RADAU and RADAU5.

TABLE II.8.3: *Reference solution at the end of the integration interval.*

| | |
|---|---|
| $y_1$ | $0.1706167732170483 \cdot 10^{1}$ |
| $y_2$ | $-0.8928097010247975 \cdot 10^{0}$ |

# References

[EP02] C. H. Edwards and D. E. Penney. *Differential Equations and Linear Algebra.* Prentice Hall, 2002.

[HW96] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II: Stiff and Differential-algebraic Problems.* Springer-Verlag, second revised edition, 1996.

[MM08] F. Mazzia and C. Magherini. *Test Set for Initial Value Problem Solvers, release 2.4.* Department of Mathematics, University of Bari and INdAM, Research Unit of Bari, February 2008. Available at http://www.dm.uniba.it/~testset.

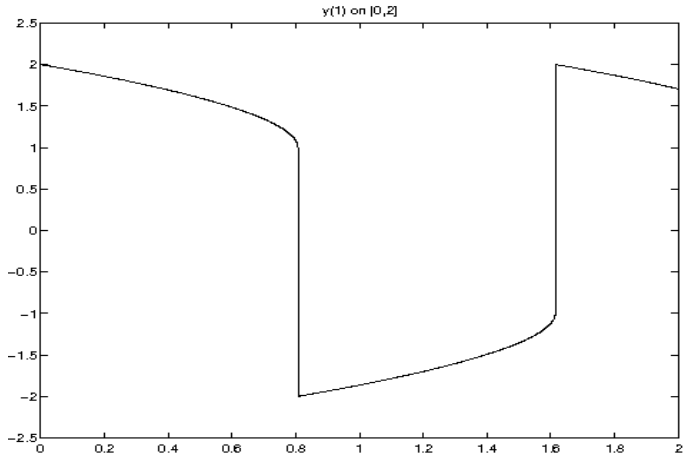


FIGURE II.8.5: *Behavior of the solution component $y_1$ over the integration interval (scaled equation)*

TABLE II.8.4: *Run characteristics.*

| solver | rtol | atol | h0 | mescd | scd | steps | accept | #f | #Jac | #LU | CPU |
|---|---|---|---|---|---|---|---|---|---|---|---|
| BIMD | $10^{-4}$ | $10^{-4}$ | $10^{-6}$ | 4.31 | 3.98 | 170 | 155 | 3684 | 151 | 170 | 0.0029 |
| | $10^{-7}$ | $10^{-7}$ | $10^{-9}$ | 9.06 | 8.73 | 301 | 293 | 7631 | 280 | 301 | 0.0059 |
| | $10^{-10}$ | $10^{-10}$ | $10^{-12}$ | 11.17 | 10.84 | 307 | 304 | 13339 | 292 | 307 | 0.0088 |
| DDASSL | $10^{-4}$ | $10^{-4}$ | | 2.89 | 2.56 | 796 | 776 | 1260 | 127 | | 0.0029 |
| | $10^{-7}$ | $10^{-7}$ | | 5.89 | 5.57 | 1943 | 1912 | 2796 | 149 | | 0.0078 |
| | $10^{-10}$ | $10^{-10}$ | | 8.96 | 8.64 | 6166 | 6110 | 7973 | 223 | | 0.0234 |
| GAMD | $10^{-4}$ | $10^{-4}$ | $10^{-6}$ | 5.40 | 5.08 | 148 | 105 | 6999 | 105 | 148 | 0.0049 |
| | $10^{-7}$ | $10^{-7}$ | $10^{-9}$ | 6.52 | 6.19 | 163 | 133 | 12727 | 131 | 163 | 0.0098 |
| | $10^{-10}$ | $10^{-10}$ | $10^{-12}$ | 10.16 | 9.84 | 244 | 216 | 18095 | 215 | 244 | 0.0137 |
| MEBDFI | $10^{-4}$ | $10^{-4}$ | $10^{-6}$ | 3.86 | 3.53 | 638 | 591 | 2179 | 92 | 92 | 0.0029 |
| | $10^{-7}$ | $10^{-7}$ | $10^{-9}$ | 6.99 | 6.67 | 1369 | 1317 | 4735 | 132 | 132 | 0.0078 |
| | $10^{-10}$ | $10^{-10}$ | $10^{-12}$ | 10.80 | 10.47 | 2862 | 2858 | 9489 | 287 | 287 | 0.0146 |
| PSIDE-1 | $10^{-4}$ | $10^{-4}$ | | 5.70 | 5.38 | 235 | 166 | 4402 | 73 | 780 | 0.0049 |
| | $10^{-7}$ | $10^{-7}$ | | 8.72 | 8.39 | 414 | 386 | 7896 | 75 | 908 | 0.0078 |
| | $10^{-10}$ | $10^{-10}$ | | 11.40 | 11.07 | 1388 | 1365 | 23066 | 131 | 1360 | 0.0224 |
| RADAU | $10^{-4}$ | $10^{-4}$ | $10^{-6}$ | 4.77 | 4.44 | 242 | 207 | 2214 | 165 | 231 | 0.0020 |
| | $10^{-7}$ | $10^{-7}$ | $10^{-9}$ | 8.28 | 7.95 | 186 | 149 | 5212 | 102 | 173 | 0.0029 |
| | $10^{-10}$ | $10^{-10}$ | $10^{-12}$ | 11.47 | 11.14 | 245 | 215 | 7589 | 148 | 224 | 0.0049 |
| VODE | $10^{-4}$ | $10^{-4}$ | | 2.93 | 2.61 | 788 | 702 | 1186 | 21 | 181 | 0.0029 |
| | $10^{-7}$ | $10^{-7}$ | | 5.65 | 5.32 | 2375 | 2200 | 3091 | 41 | 345 | 0.0088 |
| | $10^{-10}$ | $10^{-10}$ | | 8.42 | 8.09 | 6426 | 6058 | 7814 | 106 | 794 | 0.0215 |

[Sha94] Lawrence F. Shampine. *Numerical solution of ordinary differential equations.* Chapman & Hall, New York, 1994.

[vdP20] B. van der Pol. *Radio Rev.*, 1:704–754, 1920.

[vdP26] B. van der Pol. On relaxation oscillations. *Phil. Mag.*, 2:978–992, 1926. reproduced in: B.

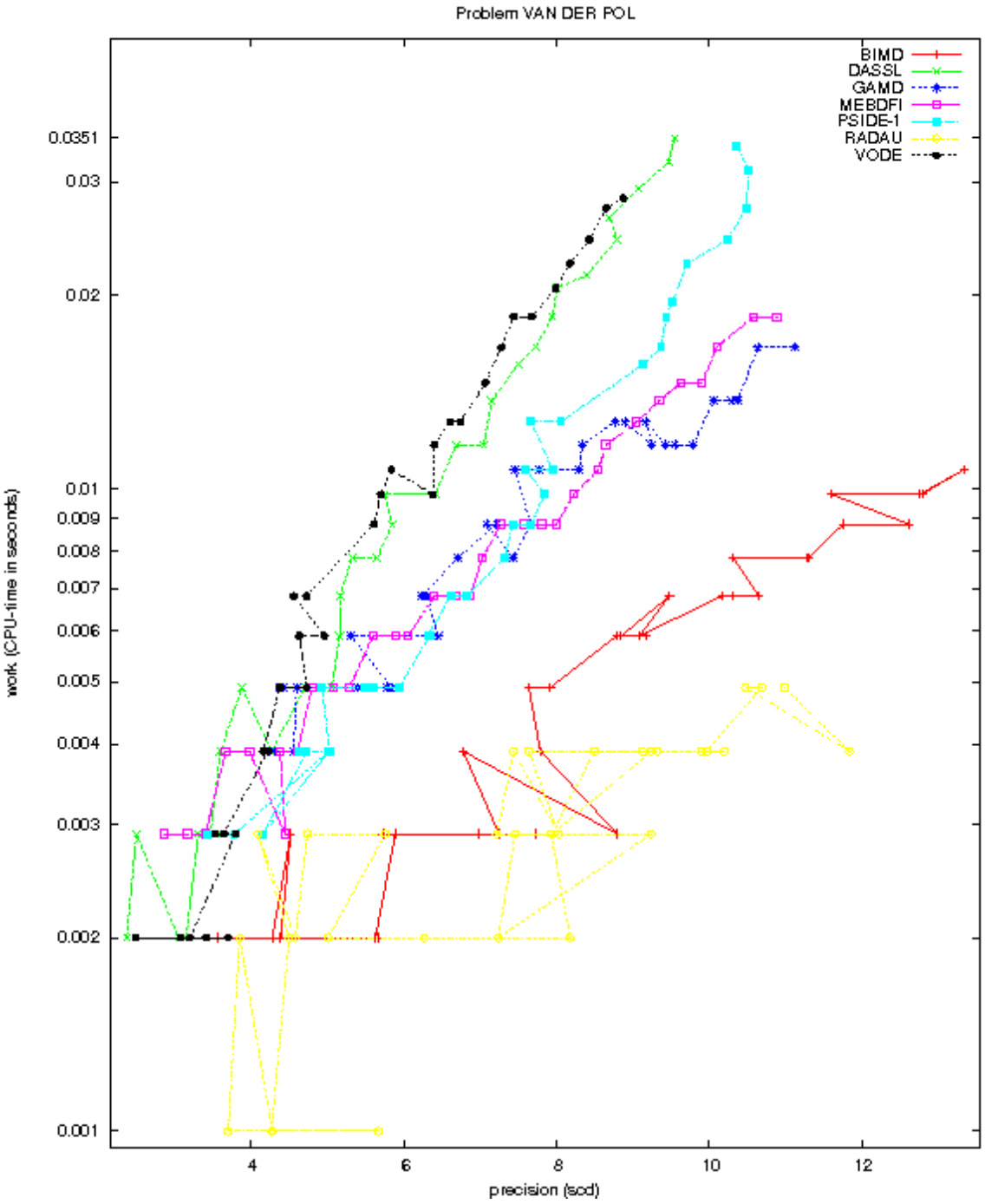


FIGURE II.8.6: *Work-precision diagram (scd versus CPU-time).*

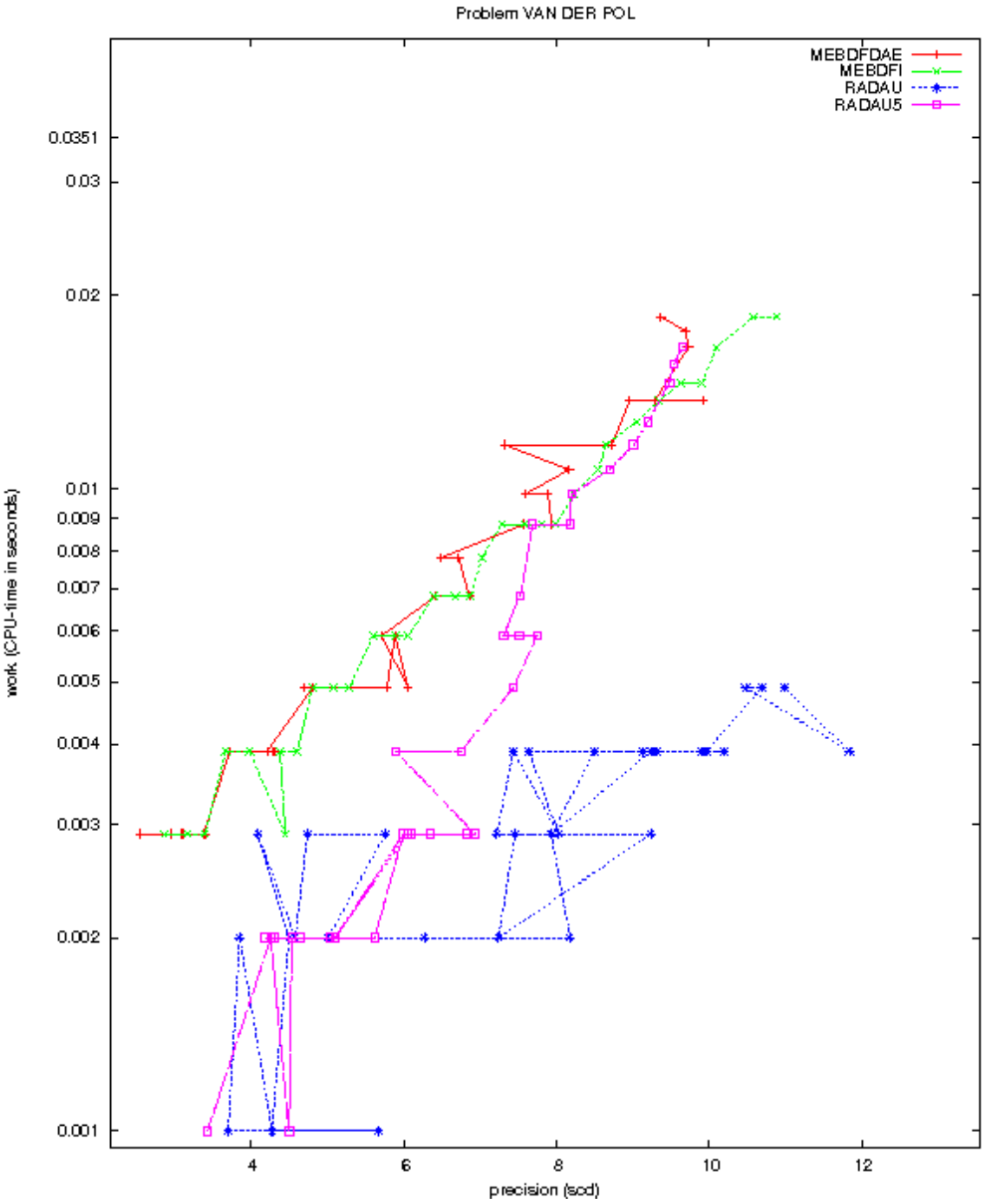


FIGURE II.8.7: *Work-precision diagram (scd versus CPU-time).*

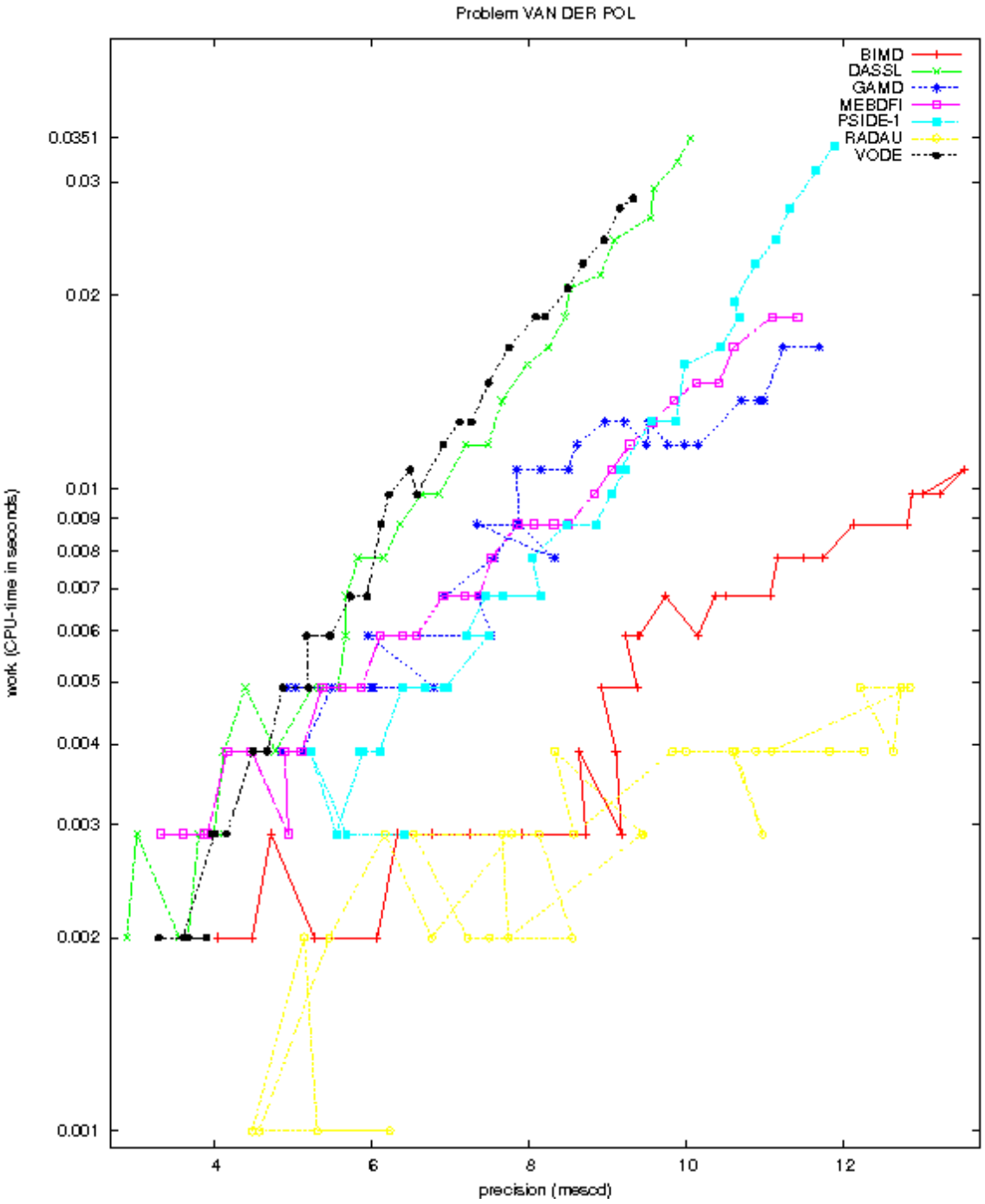


FIGURE II.8.8: *Work-precision diagram (mescd versus CPU-time).*

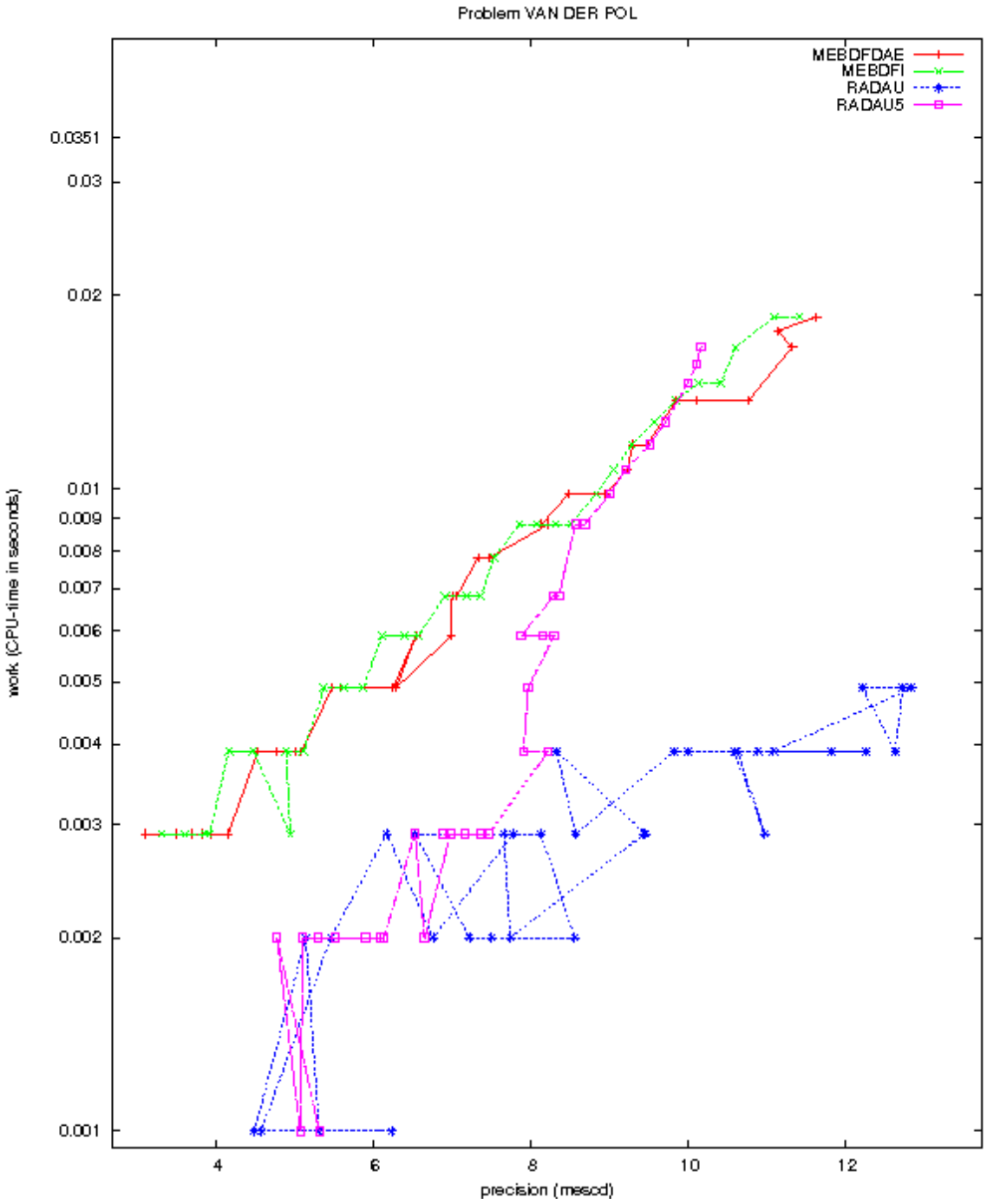


FIGURE II.8.9: *Work-precision diagram (mescd versus CPU-time).*

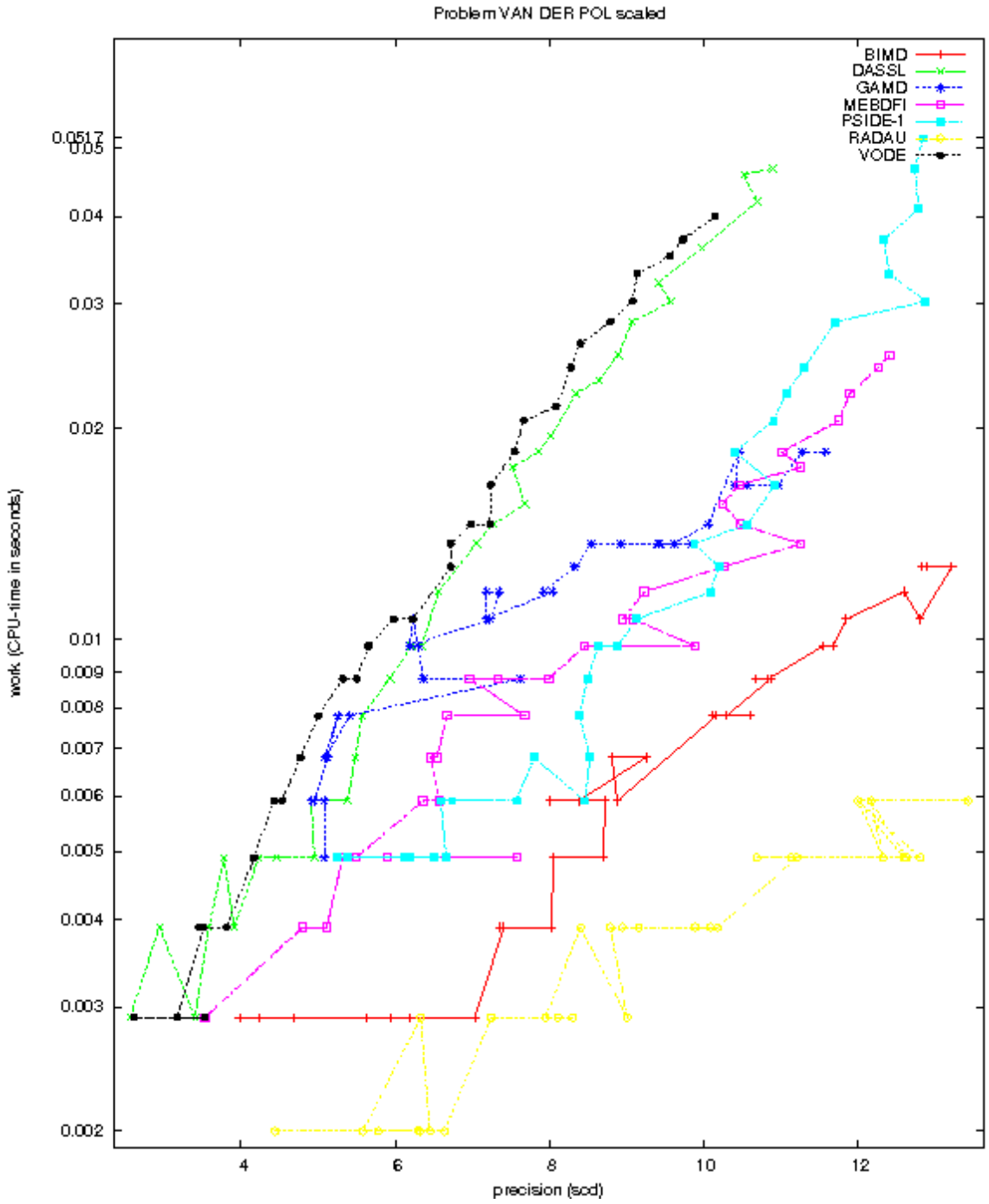


FIGURE II.8.10: *Work-precision diagram (scd versus CPU-time).*

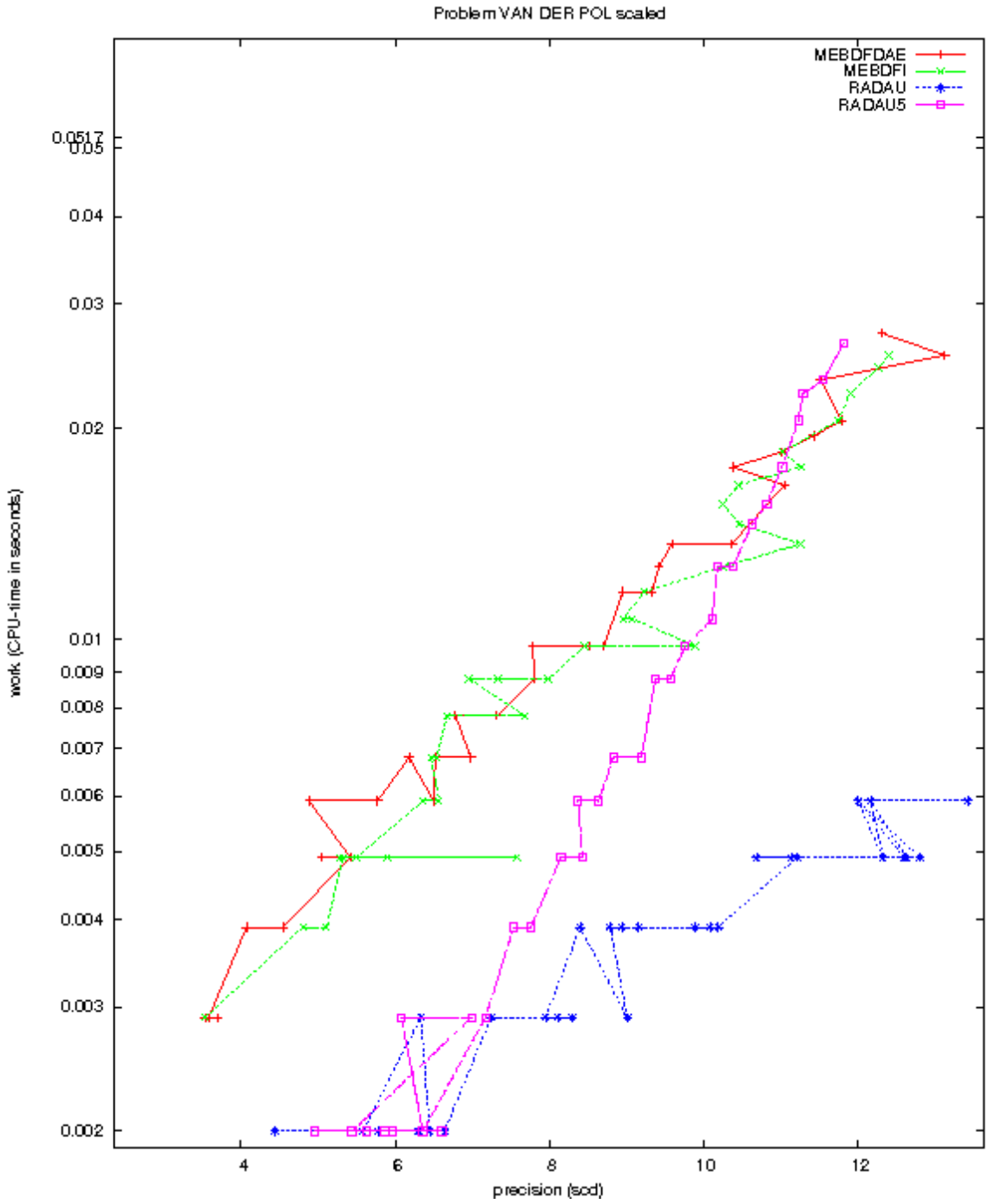


FIGURE II.8.11: *Work-precision diagram (scd versus CPU-time).*

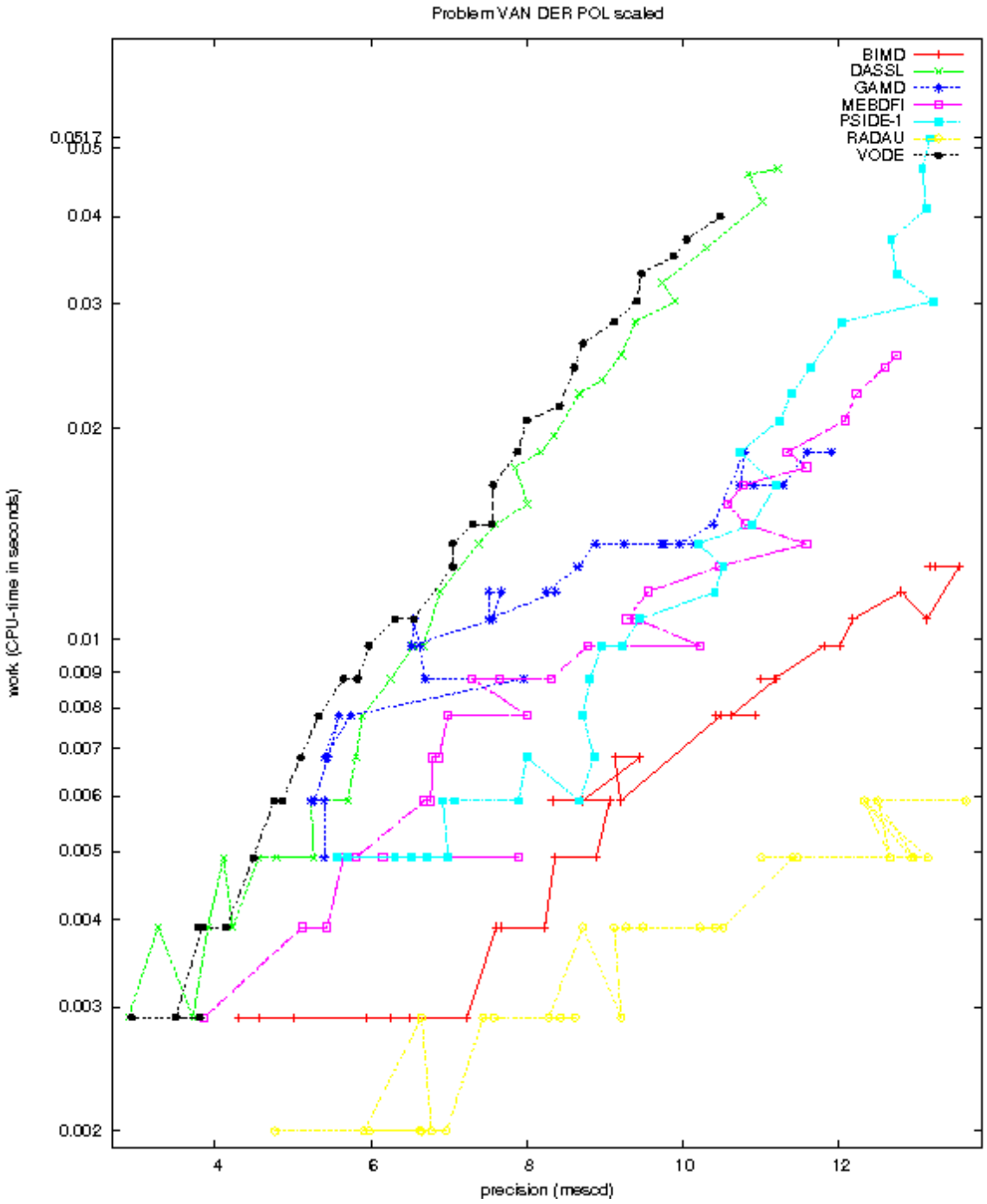


FIGURE II.8.12: *Work-precision diagram (mescd versus CPU-time).*

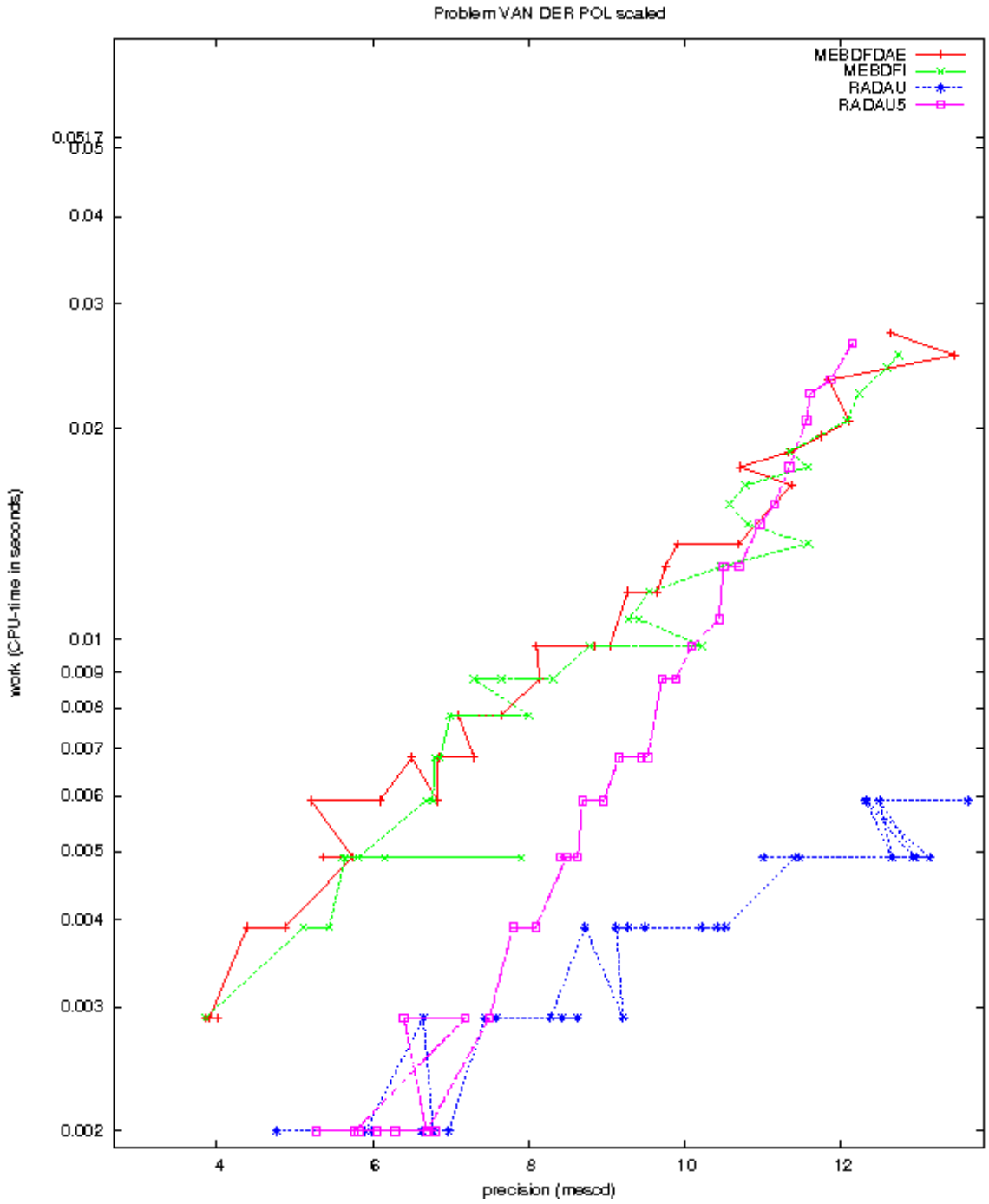


FIGURE II.8.13: *Work-precision diagram (mescd versus CPU-time).*

van der Pol, Selected Scientific Papers, vol. I, North Holland Publ. Comp. Amsterdam, 1960.