6 Pleiades problem

6.1 General information

The problem consists of a nonstiff system of 14 special second order differential equations rewritten to first order form, thus providing a nonstiff system of ordinary differential equations of dimension 28. The formulation and data have been taken from [HNW93]. E. Messina contributed this problem to the test set. Comments to `eleonora.messina@unina.it`.

The software part of the problem is in the file plei.f available at [MM08].

6.2 Mathematical description of the problem

The problem is of the form

$$z'' \quad = \quad f(z), \quad z(0) = z_0, \quad z'(0) = z'_0, \tag{II.6.1}$$

with

$$z \in \mathbb{R}^{14}, \quad 0 \leq t \leq 3.$$

Defining $z := (x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}}$, $x, y \in \mathbb{R}^7$, the function $f : \mathbb{R}^{14} \to \mathbb{R}^{14}$ is given by $f(z) = f(x,y) = (f^{(1)}(x,y)^{\mathrm{T}}, f^{(2)}(x,y)^{\mathrm{T}})^{\mathrm{T}}$, where $f^{(1,2)} : \mathbb{R}^{14} \to \mathbb{R}^7$ read

$$f_i^{(1)} = \sum_{j \neq i} m_j (x_j - x_i)/r_{ij}^{\frac{3}{2}}, \qquad f_i^{(2)} = \sum_{j \neq i} m_j (y_j - y_i)/r_{ij}^{\frac{3}{2}}, \qquad i = 1, \ldots, 7. \tag{II.6.2}$$

Here, $m_i = i$ and

$$r_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2.$$

We write this problem to first order form by defining $w = z'$, yielding a system of 28 non-linear differential equations of the form

$$\begin{pmatrix} z \\ w \end{pmatrix}' = \begin{pmatrix} w \\ f(z) \end{pmatrix} \tag{II.6.3}$$

with

$$(z^{\mathrm{T}}, w^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{28}, \quad 0 \leq t \leq 3.$$

The initial values are

$$\begin{pmatrix} z_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ x'_0 \\ y'_0 \end{pmatrix}, \quad \text{where} \quad \left\{ \begin{array}{lcl} x_0 & = & (3, 3, -1, -3, 2, -2, 2)^{\mathrm{T}}, \\ y_0 & = & (3, -3, 2, 0, 0, -4, 4)^{\mathrm{T}}, \\ x'_0 & = & (0, 0, 0, 0, 0, 1.75, -1.5)^{\mathrm{T}}, \\ y'_0 & = & (0, 0, 0, -1.25, 1, 0, 0)^{\mathrm{T}}. \end{array} \right.$$

6.3 Origin of the problem

The Pleiades problem is a celestial mechanics problem of seven stars in the plane of coordinates x_i, y_i and masses $m_i = i$ ($i = 1, \ldots, 7$). We obtain the formulation of the problem by means of some mechanical considerations. Let us consider the body i. According to the second law of Newton this star is subjected to the action

$$F_i = m_i p''_i, \tag{II.6.4}$$

where $p_i := (x_i, y_i)^{\mathrm{T}}$. On the other hand, the law of gravity states that the force working on body i implied by body j, denoted by F_{ij}, is

$$F_{ij} = g \frac{m_i \cdot m_j}{\|p_i - p_j\|_2^2} d_{ij}. \tag{II.6.5}$$

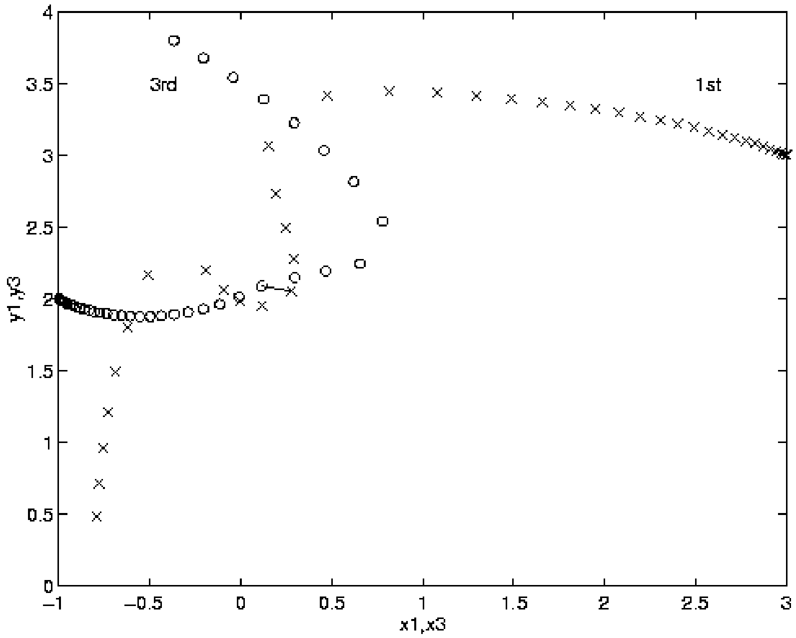

FIGURE II.6.1: *Trajectories of the first and third body on* $[0, 2]$.

TABLE II.6.1: *Quasi-collisions in Pleiades problem. The squared distance between body i and body j at $t = \tau$ is listed (values taken from [HNW93]).*

i	1	1	3	1	2	5
j	7	3	5	7	6	7
τ	1.23	1.46	1.63	1.68	1.94	2.14
$\|\|p_i - p_j\|\|_2^2$	0.0129	0.0193	0.0031	0.0011	0.1005	0.0700

Here, F_i, $F_{ij} \in \mathbb{R}^2$, g is the gravitational constant, which is assumed to be one here, and $d_{ij} = \frac{p_j - p_i}{\|p_j - p_i\|_2}$ represents the direction of the distance between the two stars. According to the principle of superposition of actions, F_i will be the sum of the interactions between body i and all the others,

$$F_i = \sum_{i \neq j} F_{ij}. \qquad \text{(II.6.6)}$$

It is easily checked that (II.6.4)–(II.6.6) and (II.6.2) are the same.

During the movement of the 7 bodies several quasi-collisions occur which are displayed in Table II.6.1. In Figure II.6.1 the behaviors of the bodies 1 and 3 in the interval $[0, 2]$ are shown; the circles and the crosses represent data obtained every 0.05 sec, the link '—' indicates the distance occurring between the two stars at $t = 1.45$.

6.4 Numerical solution of the problem

One should be aware of the fact that the Pleiades problem is a nonstiff ODE. Therefore we also include the results obtained by the nonstiff solver DOPRI5[HW96], which is based on an explicit Runge–Kutta method.

TABLE II.6.2: *Reference solution at the end of the integration interval.*

x_1	0.3706139143970502	y_1	$-0.3943437585517392 \cdot 10$
x_2	$0.3237284092057233 \cdot 10$	y_2	$-0.3271380973972550 \cdot 10$
x_3	$-0.3222559032418324 \cdot 10$	y_3	$0.5225081843456543 \cdot 10$
x_4	0.6597091455775310	y_4	$-0.2590612434977470 \cdot 10$
x_5	0.3425581707156584	y_5	$0.1198213693392275 \cdot 10$
x_6	$0.1562172101400631 \cdot 10$	y_6	-0.2429682344935824
x_7	-0.7003092922212495	y_7	$0.1091449240428980 \cdot 10$
x'_1	$0.3417003806314313 \cdot 10$	y'_1	$-0.3741244961234010 \cdot 10$
x'_2	$0.1354584501625501 \cdot 10$	y'_2	0.3773459685750630
x'_3	$-0.2590065597810775 \cdot 10$	y'_3	0.9386858869551073
x'_4	$0.2025053734714242 \cdot 10$	y'_4	0.3667922227200571
x'_5	$-0.1155815100160448 \cdot 10$	y'_5	-0.3474046353808490
x'_6	-0.8072988170223021	y'_6	$0.2344915448180937 \cdot 10$
x'_7	0.5952396354208710	y'_7	$-0.1947020434263292 \cdot 10$

Tables II.6.2–II.6.3 and Figures II.6.2–II.6.4 present the reference solution at the end of the integration interval, the run characteristics, the behavior of the solution components x_1 and y_1 over the integration interval and the work-precision diagrams, respectively. The computation of the scd values is based on the first 14 components, since they refer to the physically important quantities. The reference solution was computed on the Cray C90, using PSIDE with Cray double precision and atol = rtol = 10^{-16}. For the work-precision diagrams, we used: rtol = $10^{-(4+m/4)}$, $m = 0, 1, \ldots, 24$; atol = rtol; h0 = $10^{-2} \cdot$ rtol for BIMD, GAMD, RADAU, RADAU5 and MEBDFDAE.

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With respect to the RADAU and RADAU5 results in Table II.6.3 and Figures II.6.3–II.6.4, we remark that for generality of the test set drivers, we did not use the facility to exploit the special structure of problems of the form (II.6.3). By setting the input parameter `IWORK(9)=14`, and adjusting the Jacobian routine appropriately, RADAU and RADAU5 produces considerably better results.

These results are listed for RADAU in Table II.6.4.

References

[HNW93] E. Hairer, S.P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I: Nonstiff Problems.* Springer-Verlag, second revised edition, 1993.

[HW96] E. Hairer and G. Wanner. *DOPRI5*, April 25, 1996. Bug fix release sep 18, 1998. Available at http://www.unige.ch/~hairer/prog/nonstiff/dopri5.f.

[MM08] F. Mazzia and C. Magherini. *Test Set for Initial Value Problem Solvers, release 2.4.* Department of Mathematics, University of Bari and INdAM, Research Unit of Bari, February 2008. Available at http://www.dm.uniba.it/~testset.

TABLE II.6.3: *Run characteristics.*

solver	rtol	atol	h0	mescd	scd	steps	accept	#f	#Jac	#LU	CPU
BIMD	10^{-4}	10^{-4}	10^{-6}	2.69	2.12	113	105	1955	79	110	0.0449
	10^{-7}	10^{-7}	10^{-9}	5.38	4.81	138	127	4013	123	138	0.0888
	10^{-10}	10^{-10}	10^{-12}	8.60	8.42	154	138	6947	129	152	0.1562
DDASSL	10^{-4}	10^{-4}		0.80	0.23	428	390	589	49		0.0185
	10^{-7}	10^{-7}		3.43	3.24	1237	1224	1674	59		0.0517
	10^{-10}	10^{-10}		5.88	5.72	3778	3773	4709	61		0.1425
DOPRI5	10^{-4}	10^{-4}		1.06	0.50	100	74	602			0.0059
	10^{-7}	10^{-7}		4.06	3.49	295	244	1772			0.0176
	10^{-10}	10^{-10}		8.06	7.83	940	940	5642			0.0566
GAMD	10^{-4}	10^{-4}	10^{-6}	1.54	0.97	85	69	2751	69	85	0.0566
	10^{-7}	10^{-7}	10^{-9}	4.81	4.57	122	104	5163	104	122	0.1083
	10^{-10}	10^{-10}	10^{-12}	7.65	7.30	183	177	7927	173	183	0.1649
MEBDFI	10^{-4}	10^{-4}	10^{-6}	1.12	0.56	387	366	1339	56	56	0.0303
	10^{-7}	10^{-7}	10^{-9}	3.84	3.62	835	816	2764	86	86	0.0654
	10^{-10}	10^{-10}	10^{-12}	7.14	6.94	1868	1868	6119	189	189	0.1454
PSIDE-1	10^{-4}	10^{-4}		2.23	1.82	102	76	1710	27	364	0.0410
	10^{-7}	10^{-7}		5.26	4.70	248	223	3187	1	592	0.0712
	10^{-10}	10^{-10}		8.12	7.55	807	807	9095	1	604	0.1786
RADAU	10^{-4}	10^{-4}	10^{-6}	2.67	2.11	151	138	1053	132	151	0.0303
	10^{-7}	10^{-7}	10^{-9}	6.20	6.17	112	95	2153	83	112	0.0547
	10^{-10}	10^{-10}	10^{-12}	9.41	9.20	130	119	3001	91	130	0.0742
VODE	10^{-4}	10^{-4}		0.40	-0.17	352	325	468	6	57	0.0117
	10^{-7}	10^{-7}		2.76	2.57	1081	1043	1232	18	94	0.0303
	10^{-10}	10^{-10}		5.41	5.20	3120	3079	3351	51	203	0.0830

TABLE II.6.4: *Run characteristics obtained by RADAU with exploited special structure.*

solver	rtol	atol	h0	mescd	scd	steps	accept	#f	#Jac	#LU	CPU
RADAU	10^{-4}	10^{-4}	10^{-6}	1.72	2.11	151	138	1053	132	151	0.0234
	10^{-7}	10^{-7}	10^{-9}	5.13	6.17	112	95	2153	83	112	0.0429
	10^{-10}	10^{-10}	10^{-12}	8.27	9.20	130	119	3001	91	130	0.0586

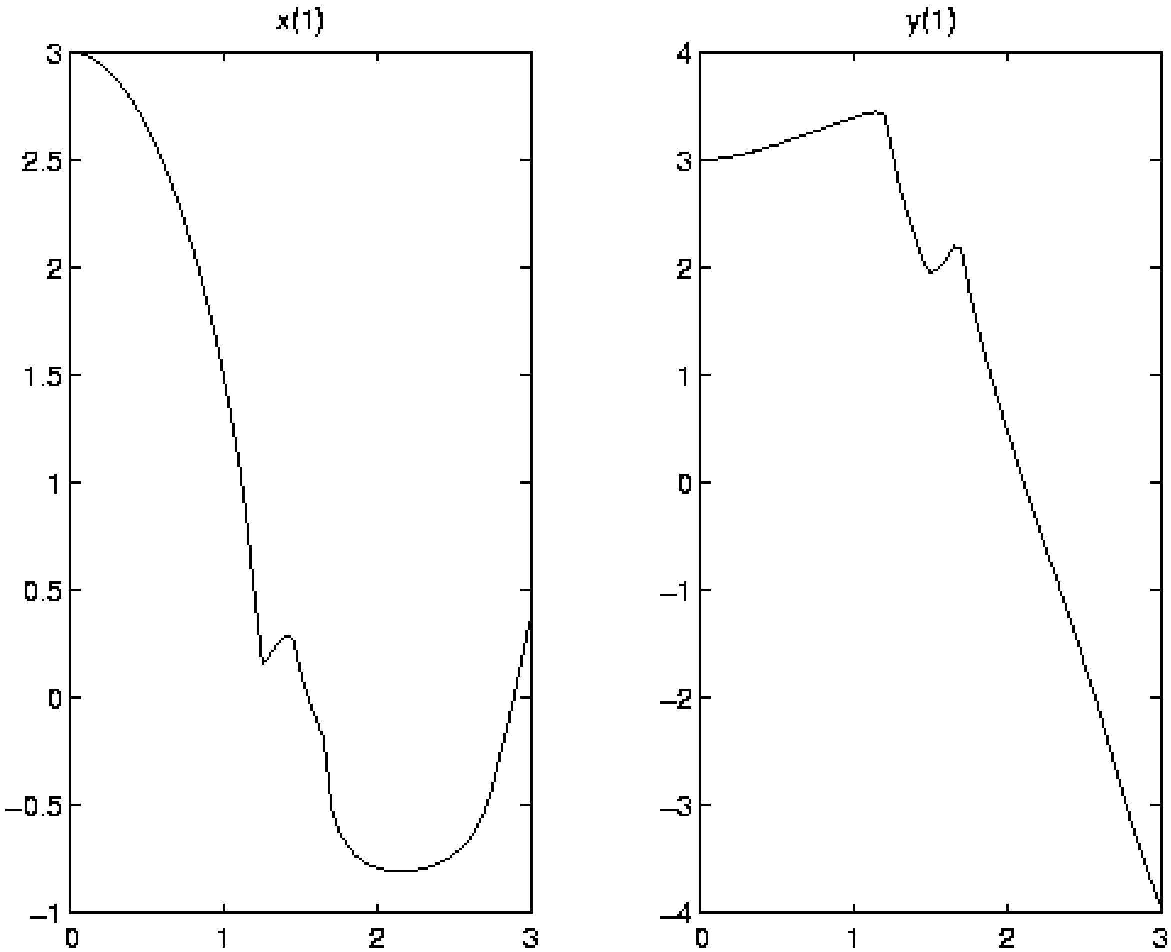

FIGURE II.6.2: *Behavior of the two solution components corresponding to the first body over the integration interval.*

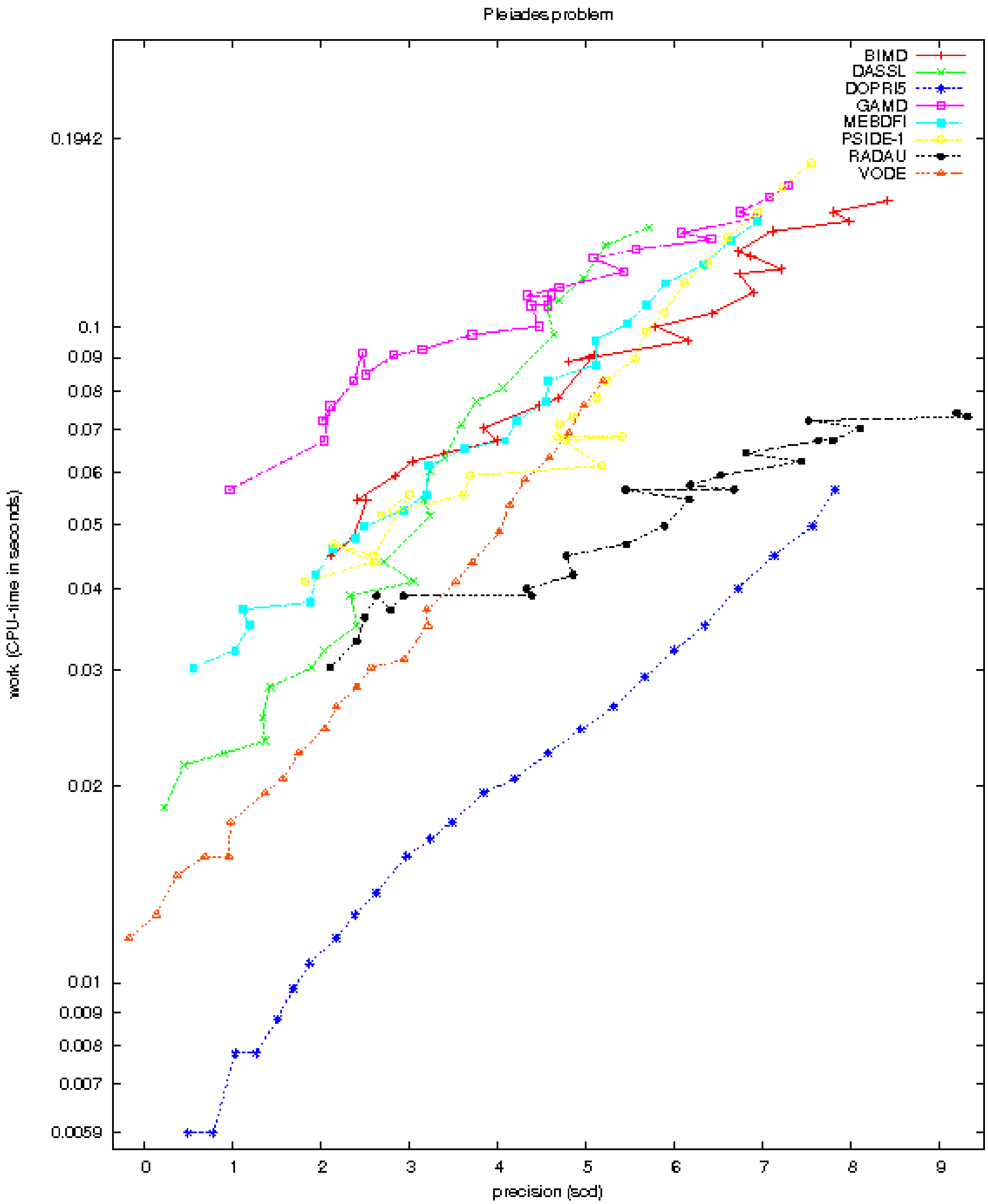

FIGURE II.6.3: *Work-precision diagram (scd versus CPU-time).*

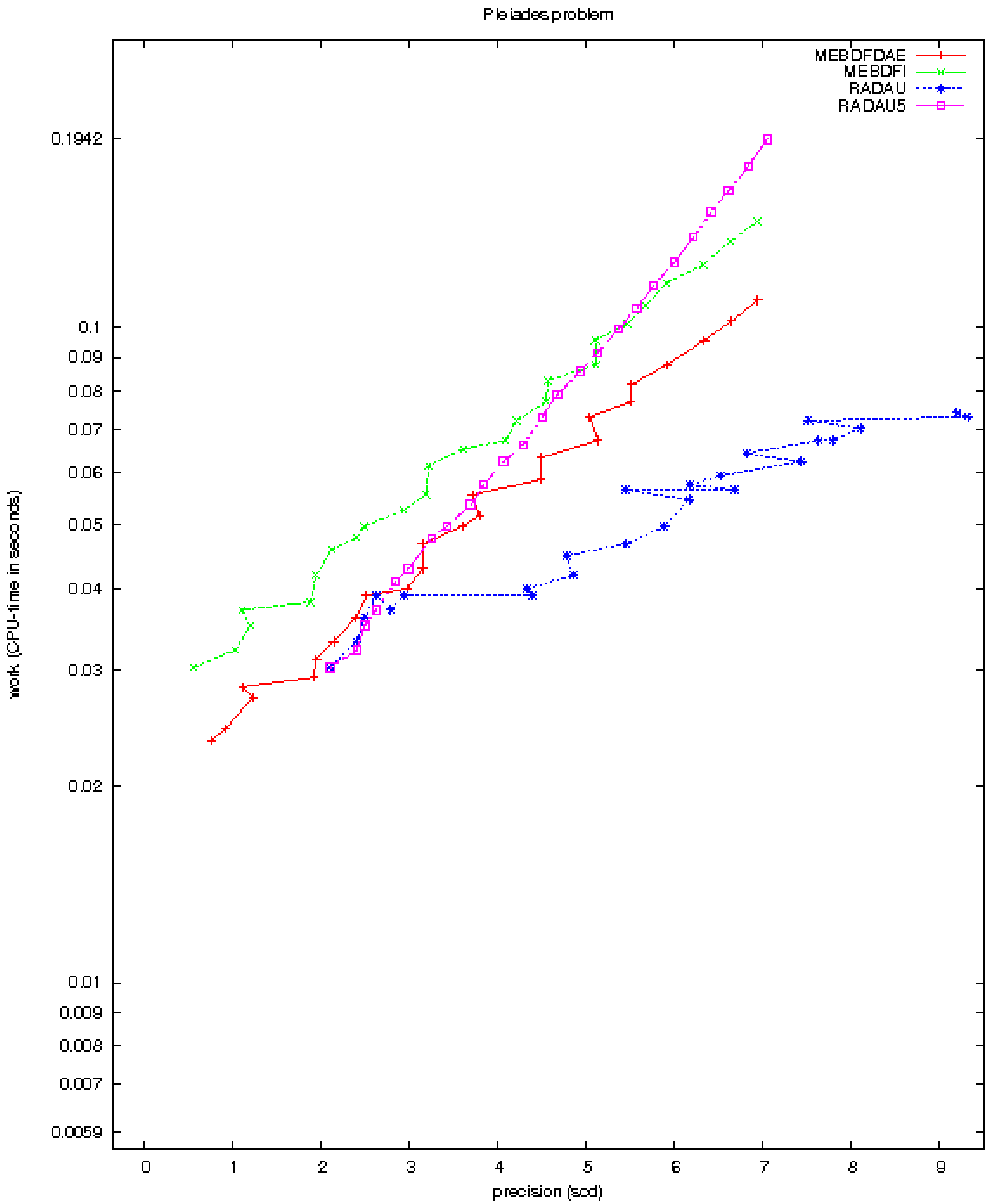

FIGURE II.6.4: *Work-precision diagram (scd versus CPU-time).*

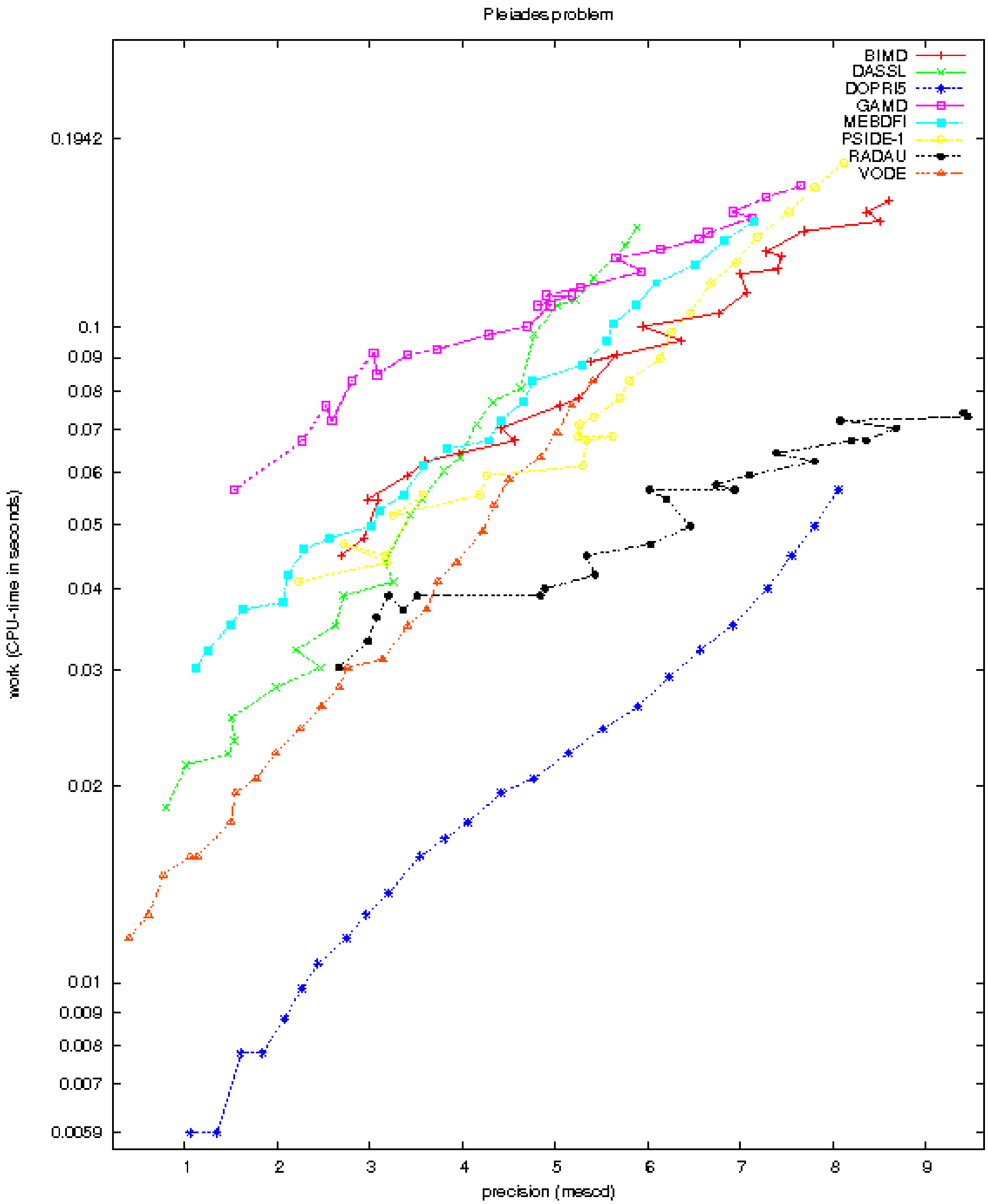

FIGURE II.6.5: *Work-precision diagram (mescd versus CPU-time).*

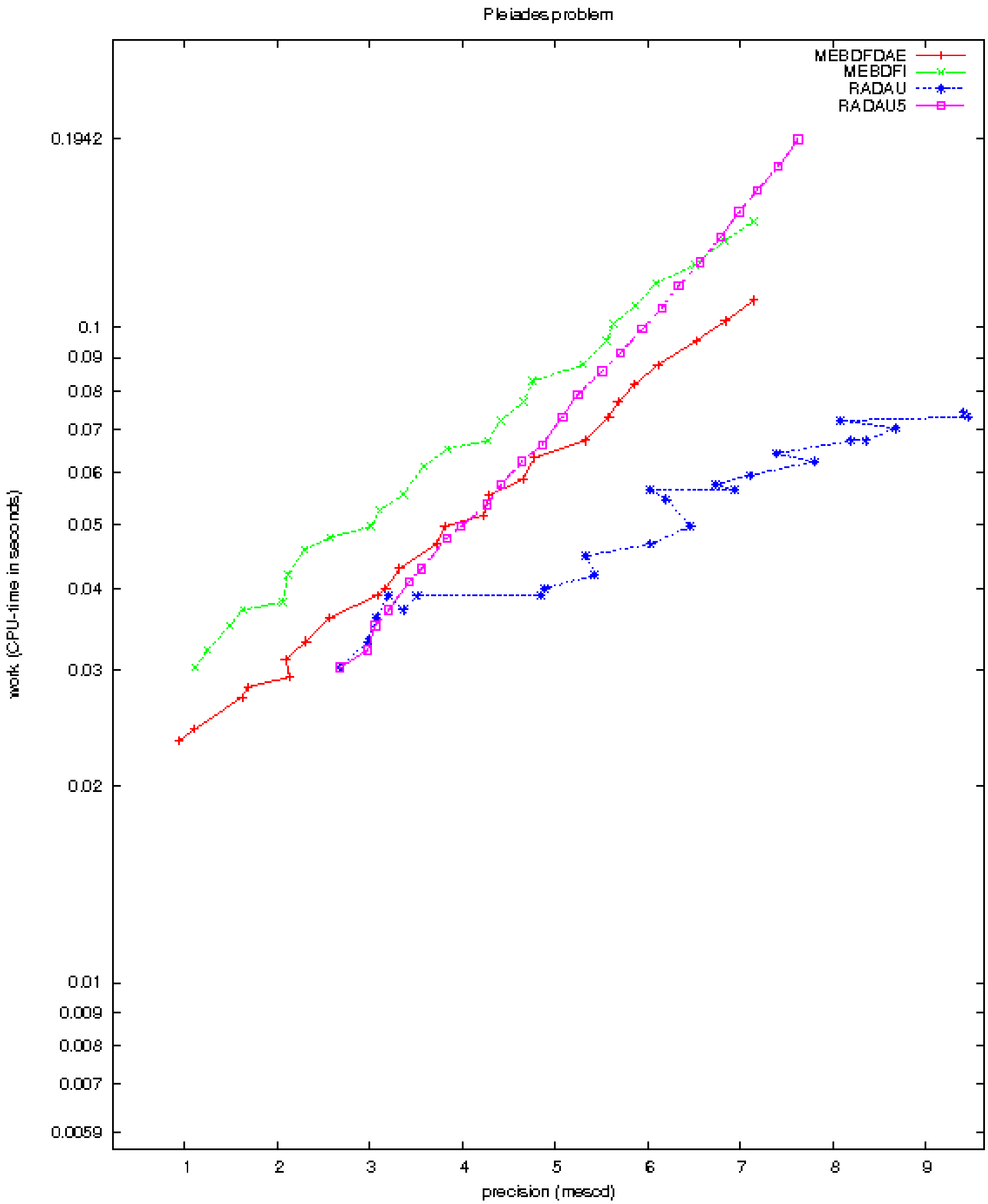

FIGURE II.6.6: *Work-precision diagram (mescd versus CPU-time).*