13 Andrews' squeezing mechanism

13.1 General information

The problem is a non-stiff second order DAE of index 3, consisting of 14 differential and 13 algebraic equations. It has been promoted as a test problem by Giles [Gil78] and Manning [Man81]. The formulation here corresponds to the one presented in Hairer & Wanner [HW96]. The parallel-IVP-algorithm group of CWI contributed this problem to the test set. The software part of the problem is in the file andrews.f available at [MM08].

13.2 Mathematical description of the problem

The problem is of the form

$$K\frac{\mathrm{d}y}{\mathrm{d}t} = \phi(y), \qquad y(0) = y_0, \quad y'(0) = y'_0, \tag{II.13.1}$$

where

$$y = \begin{pmatrix} q \\ \dot{q} \\ \ddot{q} \\ \lambda \end{pmatrix}, \quad K = \begin{bmatrix} I & O & O & O \\ O & I & O & O \\ O & O & O & O \\ O & O & O & O \end{bmatrix}, \quad \phi(y) = \begin{pmatrix} \dot{q} \\ \ddot{q} \\ M(q)\ddot{q} - f(q,\dot{q}) + G^{\mathrm{T}}(q)\lambda \\ g(q) \end{pmatrix}.$$

Here,

$$\begin{aligned} 0 &\leq t \leq 0.03, \\ q &\in \mathbb{R}^7, \\ \lambda &\in \mathbb{R}^6, \\ M &: \mathbb{R}^7 \to \mathbb{R}^{7\times 7}, \\ f &: \mathbb{R}^{14} \to \mathbb{R}^7, \\ g &: \mathbb{R}^7 \to \mathbb{R}^6, \\ G &= \frac{\partial g}{\partial q}. \end{aligned}$$

The function $M(q) = (M_{ij}(q))$ is given by:

$$\begin{aligned} M_{11}(q) &= m_1 \cdot ra^2 + m_2(rr^2 - 2da \cdot rr \cdot \cos q_2 + da^2) + I_1 + I_2, \\ M_{21}(q) &= M_{12}(q) \;=\; m_2(da^2 - da \cdot rr \cdot \cos q_2) + I_2, \\ M_{22}(q) &= m_2 \cdot da^2 + I_2, \\ M_{33}(q) &= m_3(sa^2 + sb^2) + I_3, \\ M_{44}(q) &= m_4(e - ea)^2 + I_4, \\ M_{54}(q) &= M_{45}(q) \;=\; m_4((e-ea)^2 + zt(e-ea)\sin q_4) + I_4, \\ M_{55}(q) &= m_4(zt^2 + 2zt(e-ea)\sin q_4 + (e-ea)^2) + m_5(ta^2 + tb^2) + I_4 + I_5, \\ M_{66}(q) &= m_6(zf - fa)^2 + I_6, \\ M_{76}(q) &= M_{67}(q) \;=\; m_6((zf-fa)^2 - u(zf-fa)\sin q_6) + I_6, \\ M_{77}(q) &= m_6((zf-fa)^2 - 2u(zf-fa)\sin q_6 + u^2) + m_7(ua^2 + ub^2) + I_6 + I_7, \\ M_{ij}(q) &= 0 \quad \text{for all other cases.} \end{aligned}$$

The function $f = (f_i(q, \dot{q}))$ reads:

$$
\begin{aligned}
f_1(q,\dot{q}) &= mom - m_2 \cdot da \cdot rr \cdot \dot{q}_2(\dot{q}_2 + 2\dot{q}_1)\sin q_2, \\
f_2(q,\dot{q}) &= m_2 \cdot da \cdot rr \cdot \dot{q}_1^2 \cdot \sin q_2, \\
f_3(q,\dot{q}) &= F_x(sc \cdot \cos q_3 - sd \cdot \sin q_3) + F_y(sd \cdot \cos q_3 + sc \cdot \sin q_3), \\
f_4(q,\dot{q}) &= m_4 \cdot zt(e - ea)\dot{q}_5^2 \cdot \cos q_4, \\
f_5(q,\dot{q}) &= -m_4 \cdot zt(e - ea)\dot{q}_4(\dot{q}_4 + 2\dot{q}_5)\cos q_4, \\
f_6(q,\dot{q}) &= -m_6 \cdot u(zf - fa)\dot{q}_7^2 \cdot \cos q_6, \\
f_7(q,\dot{q}) &= m_6 \cdot u(zf - fa)\dot{q}_6(\dot{q}_6 + 2\dot{q}_7)\cos q_6.
\end{aligned}
$$

F_x and F_y are defined by:

$$
\begin{aligned}
F_x &= F(xd - xc), \\
F_y &= F(yd - yc), \\
F &= -c_0(L - l_0)/L, \\
L &= \sqrt{(xd - xc)^2 + (yd - yc)^2}, \\
xd &= sd \cdot \cos q_3 + sc \cdot \sin q_3 + xb, \\
yd &= sd \cdot \sin q_3 - sc \cdot \cos q_3 + yb.
\end{aligned}
$$

The function $g = (g_i(q))$ is given by:

$$
\begin{aligned}
g_1(q) &= rr \cdot \cos q_1 - d \cdot \cos(q_1 + q_2) - ss \cdot \sin q_3 - xb, \\
g_2(q) &= rr \cdot \sin q_1 - d \cdot \sin(q_1 + q_2) + ss \cdot \cos q_3 - yb, \\
g_3(q) &= rr \cdot \cos q_1 - d \cdot \cos(q_1 + q_2) - e \cdot \sin(q_4 + q_5) - zt \cdot \cos q_5 - xa, \\
g_4(q) &= rr \cdot \sin q_1 - d \cdot \sin(q_1 + q_2) + e \cdot \cos(q_4 + q_5) - zt \cdot \sin q_5 - ya, \\
g_5(q) &= rr \cdot \cos q_1 - d \cdot \cos(q_1 + q_2) - zf \cdot \cos(q_6 + q_7) - u \cdot \sin q_7 - xa, \\
g_6(q) &= rr \cdot \sin q_1 - d \cdot \sin(q_1 + q_2) - zf \cdot \sin(q_6 + q_7) + u \cdot \cos q_7 - ya.
\end{aligned}
$$

The constants arising in these formulas are given by:

m_1	=	0.04325	I_1	=	$2.194 \cdot 10^{-6}$	ss	=	0.035
m_2	=	0.00365	I_2	=	$4.410 \cdot 10^{-7}$	sa	=	0.01874
m_3	=	0.02373	I_3	=	$5.255 \cdot 10^{-6}$	sb	=	0.01043
m_4	=	0.00706	I_4	=	$5.667 \cdot 10^{-7}$	sc	=	0.018
m_5	=	0.07050	I_5	=	$1.169 \cdot 10^{-5}$	sd	=	0.02
m_6	=	0.00706	I_6	=	$5.667 \cdot 10^{-7}$	ta	=	0.02308
m_7	=	0.05498	I_7	=	$1.912 \cdot 10^{-5}$	tb	=	0.00916
xa	=	−0.06934	d	=	0.028	u	=	0.04
ya	=	−0.00227	da	=	0.0115	ua	=	0.01228
xb	=	−0.03635	e	=	0.02	ub	=	0.00449
yb	=	0.03273	ea	=	0.01421	zf	=	0.02
xc	=	0.014	rr	=	0.007	zt	=	0.04
yc	=	0.072	ra	=	0.00092	fa	=	0.01421
c_0	=	4530	l_0	=	0.07785	mom	=	0.033

Consistent initial values are

$$y_0 = (q_0, \dot{q}_0, \ddot{q}_0, \lambda_0)^{\mathrm{T}} \text{ and } y_0' = (\dot{q}_0, \ddot{q}_0, \dddot{q}_0, \dot{\lambda}_0)^{\mathrm{T}},$$

where

$$
\begin{aligned}
q_0 &= \begin{pmatrix} -0.0617138900142764496358948458001 \\ 0 \\ 0.455279819163070380255912382449 \\ 0.222668390165885884674473185609 \\ 0.487364979543842550225598953530 \\ -0.222668390165885884674473185609 \\ 1.23054744454982119249735015568 \end{pmatrix}, \\
\dot{q}_0 &= \dddot{q}_0 = (0,0,0,0,0,0,0)^{\mathrm{T}}, \\
\ddot{q}_0 &= \begin{pmatrix} 14222.4439199541138705911625887 \\ -10666.8329399655854029433719415 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
\lambda_0 &= \begin{pmatrix} 98.5668703962410896057654982170 \\ -6.12268834425566265503114393122 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
\dot{\lambda}_0 &= (0,0,0,0,0,0)^{\mathrm{T}}.
\end{aligned}
$$

The index of the q, $\dot{q}$, $\ddot{q}$ and λ components in y is 1, 2, 3 and 3, respectively.

13.3 Origin of the problem

Formulation (II.13.1) can be rewritten as

$$
\begin{aligned}
M(q)\ddot{q} &= f(q,\dot{q}) - G^{\mathrm{T}}(q)\lambda, \\
0 &= g(q),
\end{aligned}
$$

which is the general form of a constrained mechanical system. More precisely, the problem describes the motion of 7 rigid bodies connected by joints without friction. It was promoted by [Gil78] and [Man81] as a test problem for numerical codes. [HW96, pp. 530–536] describes the system and the modeling process in full detail.

13.4 Numerical solution of the problem

The Jacobian $\partial\phi/\partial y$, needed by the numerical solver, was approximated by

$$
\begin{bmatrix} O & I & O & O \\ O & O & I & O \\ O & O & M & G^{\mathrm{T}} \\ G & O & O & O \end{bmatrix},
$$

which means that we neglect the derivatives of $f(q,\dot{q})$ as well as those of $M(q)$ and $G(q)$. Note that the evaluation of such a Jacobian does not cost anything, because M and G are already computed in the evaluation of ϕ. However, we did not exploit this in the numerical computations.

Tables II.13.2–II.13.3 and Figures II.13.1–II.13.5 present the reference solution at the end of the integration interval, the run characteristics, the behavior of the solution over the integration interval and the work-precision diagrams, respectively. In computing the scd values, only the first seven components were considered, since they refer to the physically important quantities, in computing the mescd values all the components were considered. The reference solution was computed on the Cray C90, using PSIDE with Cray double precision and atol = rtol = 10^{-14}. For the work-precision diagrams, we used: rtol = $10^{-(4+m/8)}$, $m = 0, 1, \ldots, 48$; atol = rtol; h0 = rtol for BIMD, GAMD. MEBDFDAE, MEBDFI, RADAU and RADAU5.

The failed runs are in Table II.13.1; listed are the name of the solver that failed, for which values of m this happened, and the reason for failing.

TABLE II.13.1: *Failed runs.*

solver	m	reason
GAMD	3,4,6	stepsize too small
RADAU	55, 56	stepsize too small

References

[Gil78] D.R.A. Giles. An algebraic approach to A-stable linear multistep-multiderivative integration formulas. *BIT*, 14:382–406, 1978.

[HW96] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II: Stiff and Differential-algebraic Problems*. Springer-Verlag, second revised edition, 1996.

[Man81] D.W. Manning. *A computer technique for simulating dynamic multibody systems based on dynamic formalism*. PhD thesis, Univ. Waterloo, Ontario, 1981.

[MM08] F. Mazzia and C. Magherini. *Test Set for Initial Value Problem Solvers, release 2.4*. Department of Mathematics, University of Bari and INdAM, Research Unit of Bari, February 2008. Available at http://www.dm.uniba.it/~testset.

TABLE II.13.2: *Reference solution (first 7 components) at the end of the integration interval.*

y_1	$0.1581077119629904 \cdot 10^{2}$	y_4	-0.5347301163226948	y_6	0.5347301163226948
y_2	$-0.1575637105984298 \cdot 10^{2}$	y_5	0.5244099658805304	y_7	$0.1048080741042263 \cdot 10$
y_3	$0.4082224013073101 \cdot 10^{-1}$				

TABLE II.13.3: *Run characteristics.*

solver	rtol	atol	h0	mescd	scd	steps	accept	#f	#Jac	#LU	CPU
BIMD	10^{-4}	10^{-4}	10^{-4}	0.27	3.05	46	41	1034	41	46	0.0185
	10^{-7}	10^{-7}	10^{-7}	2.82	5.38	122	122	2553	122	122	0.0459
GAMD	10^{-4}	10^{-4}	10^{-4}	0.35	2.82	82	58	2281	58	82	0.0293
	10^{-7}	10^{-7}	10^{-7}	1.53	4.54	128	116	5176	116	128	0.0693
MEBDFI	10^{-4}	10^{-4}	10^{-4}	−1.11	0.37	118	108	466	23	23	0.0078
	10^{-7}	10^{-7}	10^{-7}	1.25	3.50	300	287	1222	38	38	0.0195
PSIDE-1	10^{-4}	10^{-4}		0.22	2.95	92	75	1675	52	368	0.0410
	10^{-7}	10^{-7}		2.10	4.98	113	93	2637	63	428	0.0615
RADAU	10^{-4}	10^{-4}	10^{-4}	−0.84	1.36	96	56	810	54	96	0.0137
	10^{-7}	10^{-7}	10^{-7}	0.47	4.45	114	95	1292	90	114	0.0195

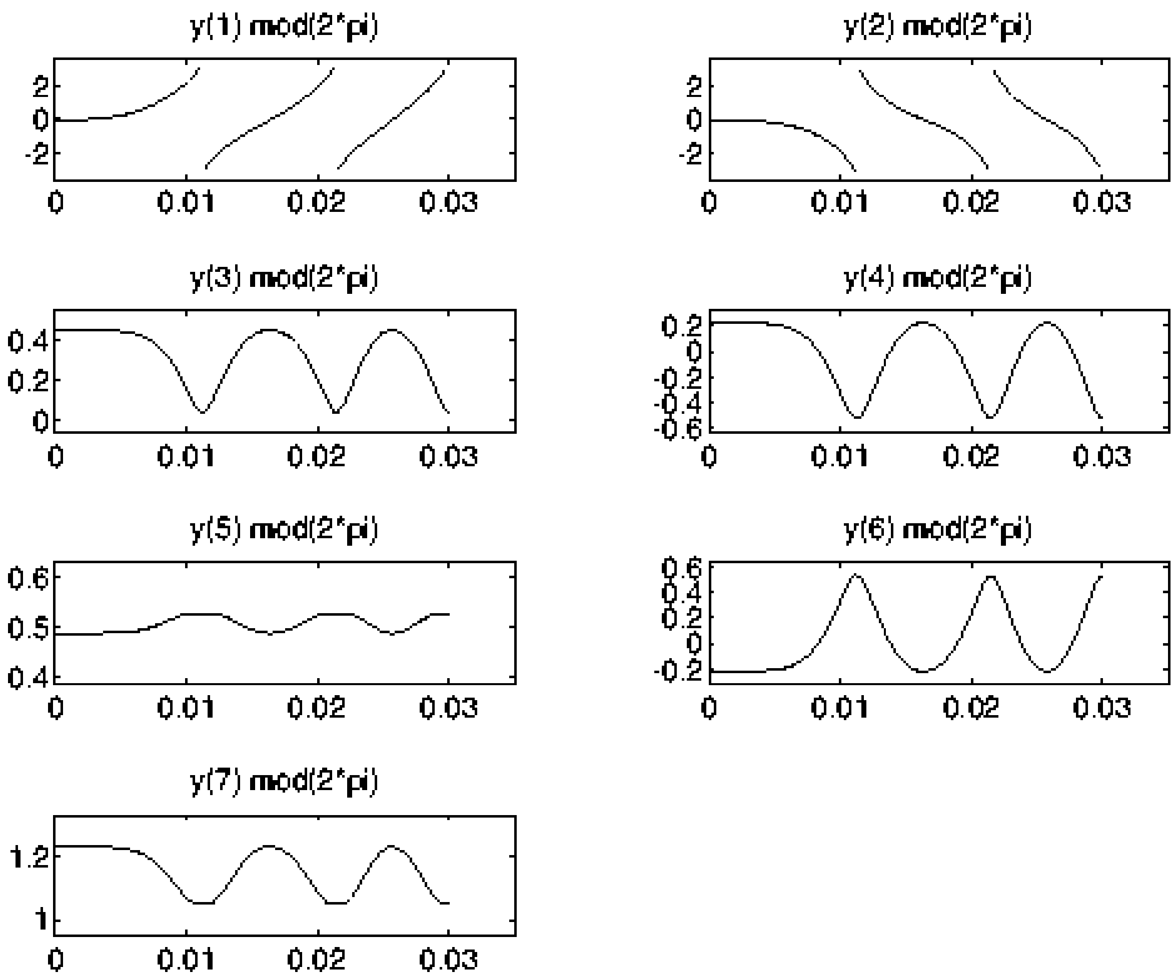

FIGURE II.13.1: *Behavior of the solution modulo 2π over the integration interval.*

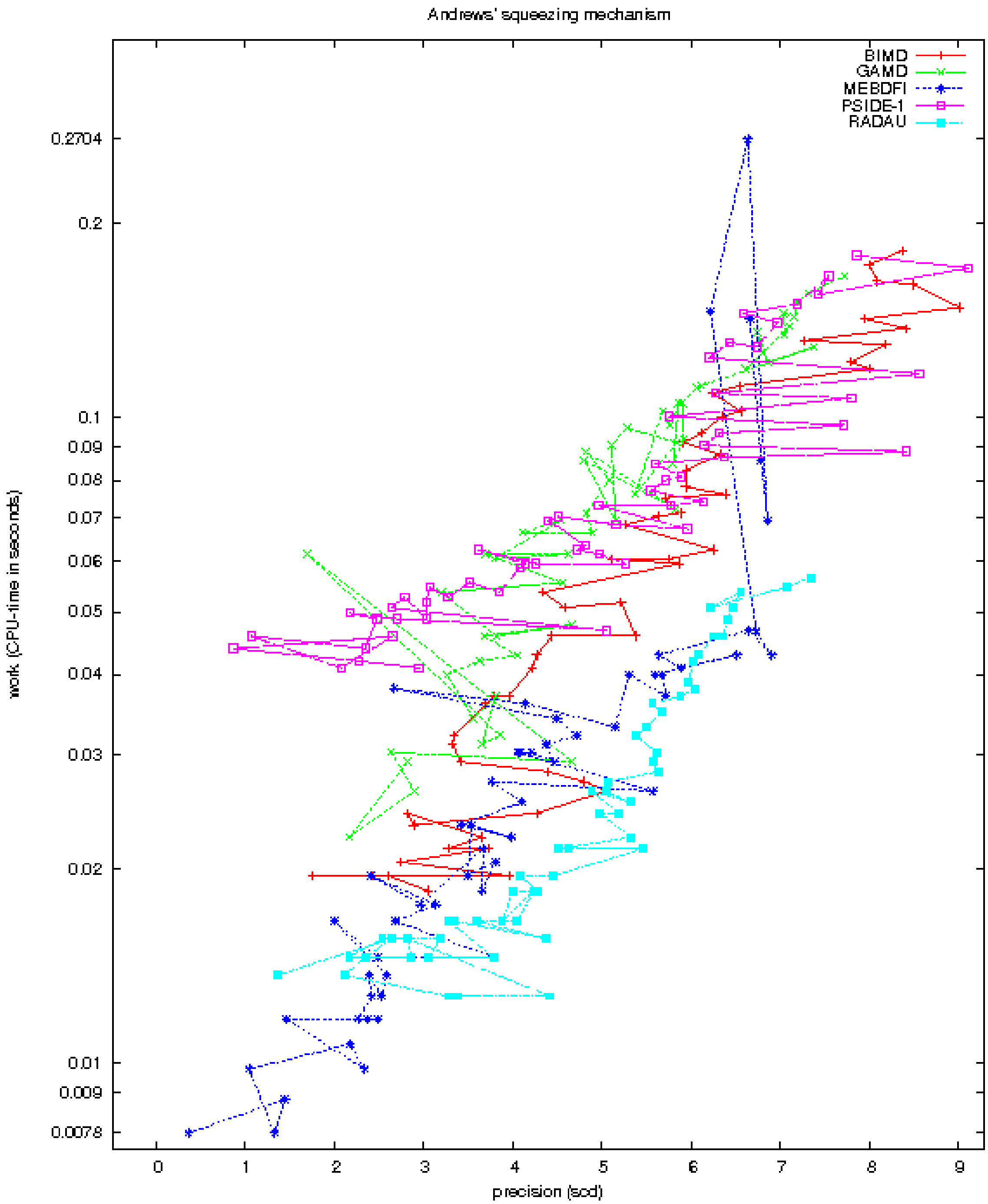

FIGURE II.13.2: *Work-precision diagram (scd versus CPU-time).*

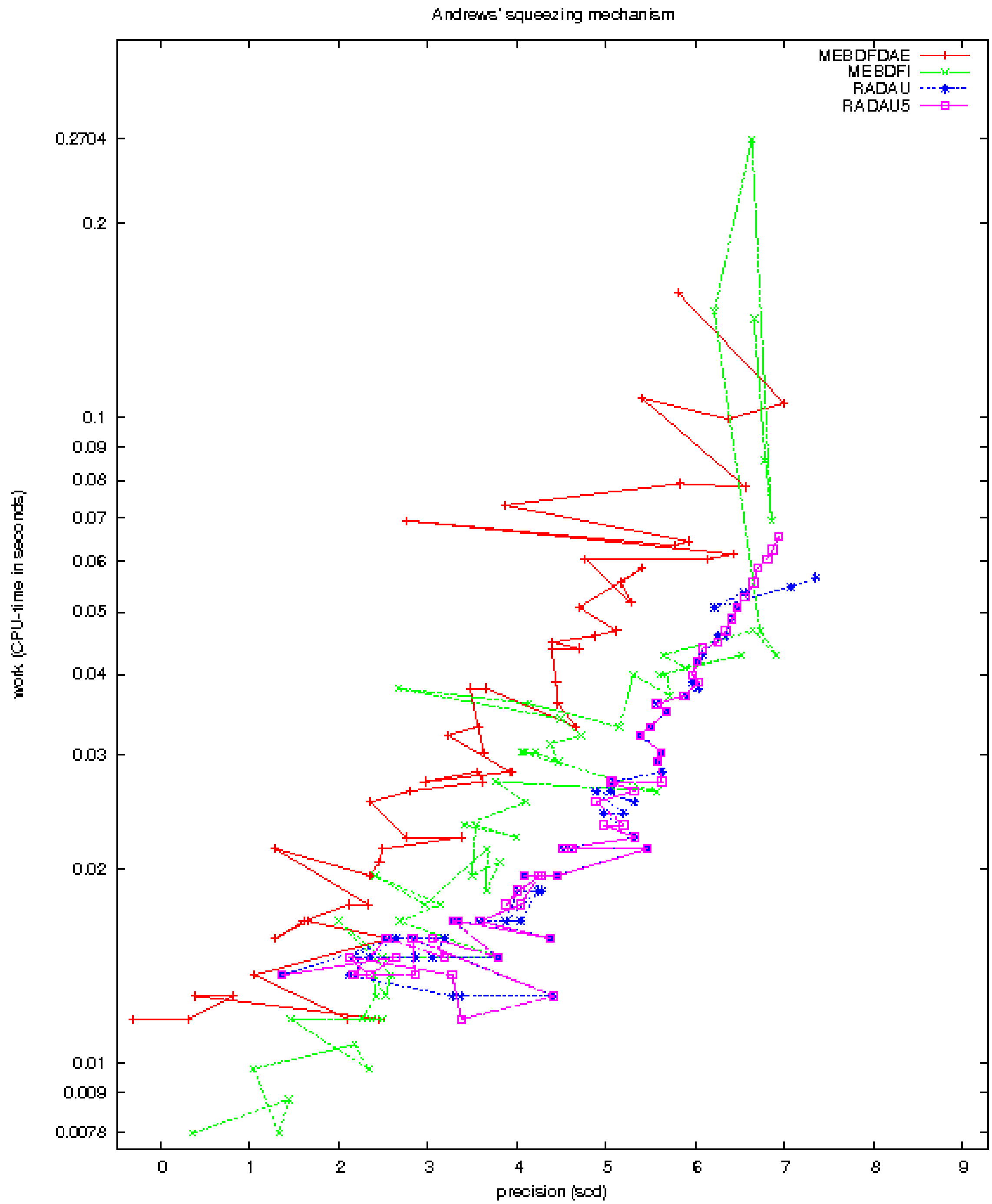

FIGURE II.13.3: *Work-precision diagram (scd versus CPU-time).*

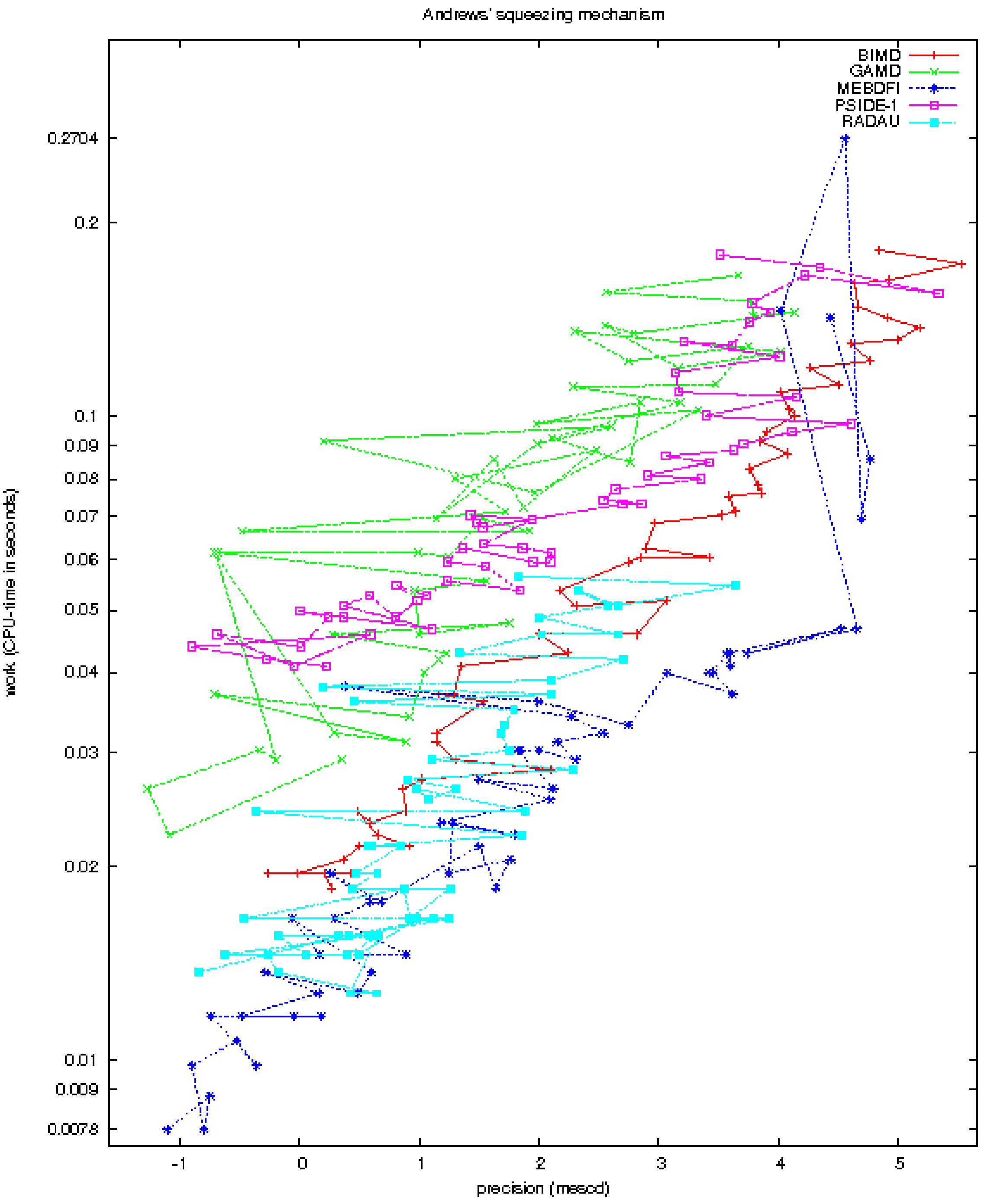

FIGURE II.13.4: *Work-precision diagram (mescd versus CPU-time).*

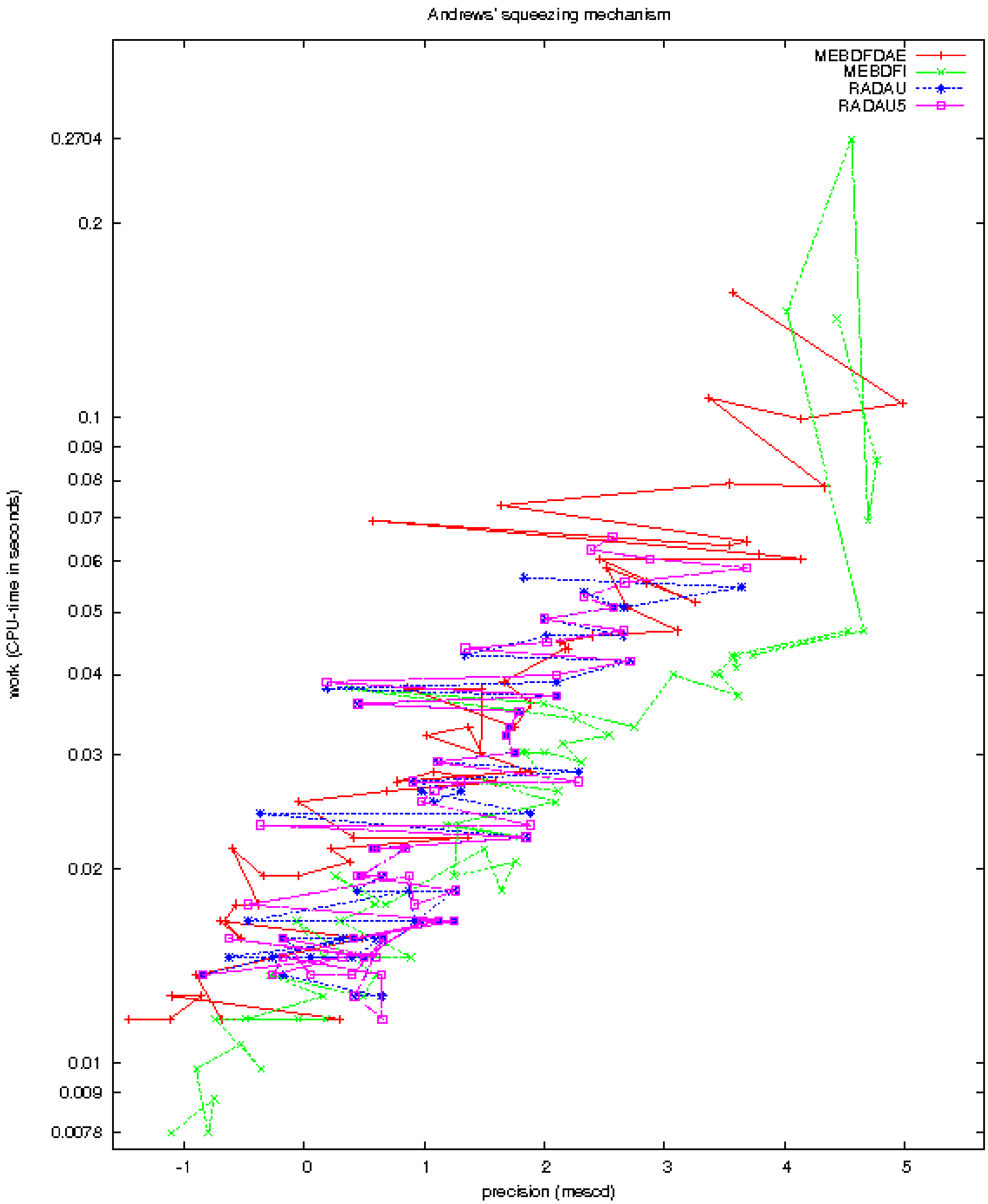

FIGURE II.13.5: *Work-precision diagram (mescd versus CPU-time).*